# Content-cum-Methodology of Teaching Mathematics

## For B.Ed. Students


राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्

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# Foreword

The National Council for Teacher Education had recommended a curriculum framework for teacher education, in which it was suggested that content-cum-methodology of teaching a subject should be taught in an integrated manner. The textbooks, that were available in the market, did not have such an integrated approach of teaching a subject. So, the Department of Teacher Education undertook to develop such textbooks.

This book Content-cum-Methodology of Teaching Mathematics for B Ed Students has been developed by a working group of the NCERT, consisting of Shri H. N. Gupta, Field Adviser, Rajasthan, Dr. V. Shankaram, RCE, Mysore; Shri N. M Badrinarayan, RCE, Mysore, Dr. K S. Khichi, RCE, Ajmer; Dr. R P. Singh, RCE, Bhopal; Dr. D.C. Sahoo, RCE, Bhubaneswar and Shri P.S Mahajan, SIE, Delhi. I am extremely thankful to all of them.

Dr. V. Shankaram and Shri H N. Gupta were responsible for editing the book. Smt. Satya Priya Gupta of the Department of Teacher Education, Special Education and Extension Services undertook the responsibility of getting the manuscript developed and of following up its printing, under the guidance of Professor R C. Das, Head of the Department. I am thankful to all of them.

I hope, the book will be useful to teacher-educators and teacher-trainees in studying content-cum-methodology of teaching mathematics at the secondary teacher education level. The Council would welcome suggestions for the improvement of the book.

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# Preface

This book is an attempt to integrate content of mathematics with methodology of teaching. For this purpose, some important topics from the secondary schools syllabus in mathematics as recommended by the NCERT have been chosen and these have been dealt with thoroughly, both with regard to clarification of the concepts and principles, and the methodology suitable for teaching these concepts and principles.

In each topic, the main concepts and sub-concepts are first discussed in detail, and the necessary related higher knowledge required to clarify these concepts is also given. The objectives of teaching this topic to the secondary school students are stated and the appropriate teaching strategies to realise these objectives have been discussed. At the end, some assignments have been given for the teacher-trainees, who are expected to use this book.

The first chapter describes the nature and scope of mathematics. It gives a general idea of the nature of mathematics, its early development and the mathematical methods like induction and deduction that are used in the solving of mathematical problems. The remaining chapters deal with the content and methodology in an integrated manner. Topics have been chosen from the area of commercial mathematics, statistics, trigonometry, geometry and algebra and number system. These topics should be taken as representative of the secondary school syllabus and not exhaustive of the same. It is hoped that a study of these topics together with the teaching strategies appropriate to them will also enable the teacher trainee to develop appropriate teaching strategies for other topics in the secondary school syllabus.

It is also hoped that this book will fill a felt-need in the secondary teacher education curriculum. We would appreciate any suggestions from teacher-educators and teacher-trainees for its improvement.

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# Contents

CHAPTER 1

# The Nature and Scope of Mathematics

## Introduction

The earliest recorded history reveals that some familiarity with mathematics has been regarded as an undisputable tool of practical utility and also an essential element of intellectual equipment of every cultured person. All mathematical developments have roots in more or less practical needs, but gradually mathematics shapes itself into a postulational deductive system. It is, therefore, not easy to define mathematics with certainty in precise words. The various definitions (available) throw light on some important aspects only. Some definitions of mathematics as given by eminent mathematicians are as follows :

(*a*) "Mathematics is the science which draws necessary conclusions"—Benjamin Peirce.

(*b*) "The mathematicians...reason correctly, but only when everything has been explained to them in terms of definitions and principles..."—B. Pascal.

(*c*) "Mathematics is the giving of the same name to different things"—H. Poincare.

(*d*) "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true"—B. Russel.

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Q. 1. Do you think that the definitions of mathematics as given by the eminent mathematicians give a full description of mathematics ? If not, list some of their shortcomings

Q. 2. Try to give a definition of mathematics, basing it on school mathematics (algebra, arithmetic and geometry).

Q. 3. Do you agree with the view that algebra is generalised arithmetic ? Give reasons in support of your answer.

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**Motivation for the Development of Mathematics—a Historical Perspective**

Though it is very difficult to say at what time of history and in which part of the world mathematics had its birth, yet there are certain important footprints on the sands of time which give one clues about the discovery and development of mathematics from time to time. Let us introduce you to some of these important landmarks in the early history of mathematics.

History has, on its record, the thesis that mathematics arose from necessity. The annual inundations of the Nile Valley forced the Egyptians to develop some means of redetermining land markings. Thus the word Geometry had its genesis in 'measurement of the earth' (Geo=the earth, metrin=to measure). The Babylonians, likewise, encountered an urgent need for mathematics in the construction of the great engineering structures Similar undertakings undoubtedly occurred in early times in South Central Asia along the Indus and Ganges rivers and in Eastern Asia. Engineering, financing, irrigation, flood control, navigation and administration of such projects required mathematics Again, a usable calendar had to be developed to serve the agricultural needs. The demand for some system of uniformity in barter was present even in the earliest civilizations, which provided a further stimulus to the development of mathematics, beyond that implied by primitive counting, originated during 5th, 4th and 3rd centuries B C. in the ancient orient as a practical science to assist in agriculture, engineering and business. While the initial stress was on mensuration and practical arithmetic, that skills should develop for application, instruction and development of the science was natural and this in turn helped tendencies towards abstraction and study of the subject, to some extent, for its own sake. In this way, a basis for the beginning of theoretical geometry arose out of mensuration and the traces of elementary algebra grew from practical arithmetic. Considering the nature (rather than the content) of the early pre-Helenistic mathematics, it is to be noted that the mathematical relations employed by the Egyptians and Babylonians resulted essentially from trial and error methods. That is to say, the earliest mathematics was little more than a practically workable empiricism—thumb-rule procedures that gave results of sufficient acceptability for simple needs of those early civilisations. For instance, an Egyptian formula for the area of a circle was to take eight-ninth of the diameter and this is equivalent to taking $\pi\ (\frac{4}{3})^4$ which is obviously incorrect. It would suffice to say that simple empirical reasoning (also known as inductive reasoning)—formulation of conclusions based upon experience and observation, devoid of understanding and logical element—formed the essence of early mathematics. This involved good guessing and heavy reliance on intuition and considerable experimentation.

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Q. 1. In ancient times people used hand and fingers to measure length. 1 hand=24 fingers. It is given in *Sulva Sutra* that the area of the square of side "2 hands" is equal to the area of a circle of radius one hand and three fingers Taking this into consideration, find the value of $\pi$ and compare it as given in the Egyptian formula for the area of the circle.

Q. 2. One of the values used for $\pi$ by ancients is $\left(\frac{4}{3}\right)^4$. Is this value more accurate than $\frac{22}{7}$ and $\sqrt{10}$, which are also different approximations of $\pi$ ?

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**Indian Contribution to Mathematics**

The Hindu civilisation, which began in the valley of the Indus river in India, dates back to the early days of Egypt and Mesopotomia Hindu mathematics was developed to serve astronomy and so it emphasized arithmetic. Hindus climbed lofty heights, but their paths are no longer traceable. They were in the habit of putting all mathematical results into verses and clothing them in obscure and mystic language and were not in the habit of preserving proofs. While the Hindu mind was arithmetical (number), the Greek mind was predominantly geometrical (form) Numerical symbolism, the science of numbers and algebra attained far greater perfection in India than they had previously reached in Greece. The history of Hindu Mathematics may be divided into five periods :

1. Ancient period (3000 B.C. to 500 B.C.)
2. Early period or Dark period (500 B.C. to A D. 500)
3. Medieval period or Golden period (A.D. 500 to A.D. 1200)
4. Post-medieval period (A.D. 1200 to A.D. 1800)
5. Present period (from A D. 1800 onwards).

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Q. Is it possible to construct a square altar and a circular altar which are equal in area ? Does our geometrical method permit us to do so ? Support your answer with reasons
In constructing such altars the ancient people used mustard seeds. They thought if the same quantity of mustard seeds covered both the altars, they would be equal in area. How far were they right in thinking so ?

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## 1. ANCIENT PERIOD (3000 B.C TO 500 B.C)

Ancient period may be further sub-divided into three main periods :

(i) *Vedic period (3000 B C. to 1000 B.C)* . In this period, the number system and the decimal system were invented. During this period, various interpretations were derived from Vedas and put into the form of Samhitas (संहिता). At one place in the thirteenth Samhita (6/2,4,5) they have given the following relationship .

$$39^2 = 36^2+25^2$$

This is a particular case of Pythagoras theorem.

(ii) *Sulva period (1000 B.C. to 500 B.C)* The knowledge of geometry was one of the important features of this period. It was the belief of Aryans that yajna is the only means for the attainment of supreme life. Yajna was performed at different geometrical altars. The yajna was useful only when performed during the specified period at the specified altar This was not possible without the knowledge of mathematics The altar requires the knowledge of geometry. Different geometrical theorems are stated in *Sulva Sutra*. Sulva was considered as the rule of cord used in the construction of altars. The knowledge of Pythagoras theorem was also prevalent during this period. This is stated by Baudhayan (1000 B.C.).

(iii) *Vedanta period (1000 B.C to 500 B.C.)* . This period is famous for astronomical work. Astronomy, as a branch of mathematics, studies the positions and movements of stars and planets

## 2. EARLY PERIOD (500 B.C. to A D. 500)

Only Jain religion *Grantha Book* and some pages of *Bakhshali Ganit* of this period are available at present. The following inventions were made in this period :

(*a*) Place value system of writing numbers.
(*b*) Invention of zero.
(*c*) Introduction of algebra.
(*d*) Development of arithmetic (Bakhshali Ganit).
(*e*) Development of astronomy and Surya Siddhanta.

In the sixth century, Varaha Mihira Acharya wrote his *Pancha Siddhantika* which gives a summary of *Surya Siddhanta* and four other astronomical works, then in use. The concepts of Interest and Per cent were also known in this period.

## 3. MEDIEVAL PERIOD (A.D. 500 to A.D. 1200)

This period starts from Aryabhata and continues up to Bhaskar II.

Aryabhata has consolidated the rules of mathematics into 33 verses in his book *Aryabhatiya*.

1. Rule for calculating the square root of a number

   भागहरेद वर्गान्नित्य द्विगुणेन वर्गमूलेन ।
   वर्गाद्वर्गे शुद्धे लब्ध स्थानान्तरे मूलम् ।।

2. Rule of three ·

   त्रैराशिक फल राशि तमथेच्छा राशिना हतं कृत्व ।
   लब्ध प्रमाण भजिते तस्मादिच्छा फलमिदं स्यात ।।

Aryabhata is considered a pioneer in the field of algebra. He gave the value of $\pi$ as $\frac{62832}{20,000}$ which is correct upto fourth decimal. He stated that the earth is moving while the stars are stationary. This is described in "Aryabhatiya Goal-pad".

In the 5th century an anonymous Hindu astronomical work was known as Surya Siddhanta which is regarded as a standard work. Brahmagupta wrote *Brahma-Sutra-Siddhanta*—the revised system of Brahma It is believed that it is Brahmagupta who gave zero its status. He has given a concept of infinity in this book. According to him, if a positive or a negative number is divided by zero the quotient will be infinity. He wrote श्वोद्धतमृणं धनवा तच्छेदम्. He also gave the formulae for finding out the volumes of the prism, cone and pyramid.

$$\text{Volume of pyramid} = \frac{H}{3}\left[\triangle + \triangle' + \sqrt{\triangle\triangle'}\right]$$

where $H$ is the height, $\triangle$ is the area of base and $\triangle'$ is the area of other surfaces.

He also gave the formula to find the sum of geometric series. If $a$ is the first term, $r$ is the common ratio and $n$ is the number of terms, then

$$S = a + ar + ar^2 + \ldots n \text{ terms} = \frac{a(r^n - 1)}{r - 1}$$

He also stated the relationship between the two intersecting chords of a circle. If $AB$ and $CD$ are two chords intersecting at $E$, the mid-point of $CD$, then

$$4\,AE \times BE = (CD)^2$$

This relationship is also described in the Euclidean Geometry. Only the chapter of Kuttaka (Indeterminate equation) of *Brahma-Sutra Siddhanta* is available and due to this method algebra was known as Kuttaka Ganit.

He clearly stated the relationship between the sides of the right angled triangle in one of his verses.

कर्णकृते: कोटकृति विशोध्यमूल भुजो भुजस्य कृतिम् ।
प्रोह्द पद कोटि: कोटि वाहु कृति युति पदं कर्ण: ॥

i.e कर्ण$^2$ — कोटि$^2$ = भुज$^2$
कोटि$^2$ + भुज$^2$ = कर्ण$^2$

This statement was recognised as Pythagoras Theorem. Mahavira belonged to 9th century and wrote *Ganit Sar-Sangrah*. He gave the method of finding LCM, which is used even now. He also gave the formula for finding the area of a quadrilateral.

The area of a quadrilateral $= \sqrt{S(S-a)\,(S-b)\,(S-c)\,(S-d)}$ where $a,b,c,d$ are the sides and $S$ is the semi-perimeter of the quadrilateral

This formula is true only for the cyclic quadrilaterals. Mahavira has also given the formula for calculating the area of an ellipse. Sridhar Acharya wrote three books—*Trisanthika, Patiganit*, and Algebra. He gave a method of solving quadratic equations. He also gave the geometrical interpretation of algebraic formula. Sripati wrote two books—*Siddhanta-Shekhar* and *Ganit-Tilak*. After Sripati, Bhaskaracharya's contribution is notable. A work entitled *Siddhanta-Siromani* was written by Bhaskaracharya (A.D. 1150). The two important chapters in this book are—the Lilavathi (the beautiful) and Vija-ganita (rod-extraction) devoted to arithmetic and algebra. Lilavathi was the lone daughter of Bhaskara and it is said that Bhaskara decided to teach his daughter, who became a widow at a very young age, to keep her mentally alert and occupied. The text is believed to be devoted to her. It contains problems stated in the form of verses in Devnagari script, on arithmetic, algebra and geometry and are posed to the intelligent. An example is stated below for the benefit of the reader.

अलिकुल दलमूलं मालती यातमषटौ
निखिल नवम् भागश्चलिनी भृंगमेकं
निषि परिमल लुब्धं पद्ममध्ये निरुद्धं
प्रतिरण तिरणंत ब्रूहिकांतेऽलि संख्या

*Meaning* . Out of a swarm of bees, a number equal to the square root of half their number went to the Malati flowers, $\frac{8}{9}$ ths of the total number also went to the same place. A male bee enticed by the fragrance of the lotus got into it. But when it was inside it, night fell, the lotus closed, and the bee was caught inside. To its buzz, its consort was replying from outside.

What is the number of bees ?

For the solution of the problem, we have the equation

$$\sqrt{\frac{x}{2}} + \frac{8}{9}x + 2 = 2 \quad \therefore x = 72$$

He has corrected the concept of zero as laid down by Mahavira and Brahmagupta

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Q.1. List five mathematics books written by ancient Indians Also name the authors.

Q 2. Solve the following problems from *Lilavathi* :

(*i*) Out of a party of monkeys, the square of 1/5th of their numbers diminished by three went into a cave The remaining one climbed up a tree. What is the total number of monkeys ?

(*ii*) On a pillar 9 cubits high is perched a peacock. For a distance of 27 cubits, a snake is coming to its hole at the bottom of the pillar. Seeing the snake, the peacock pounces upon it. If their speeds are equal, tell me quickly at what distance from the hole is the snake caught.

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## 4. POST-MEDIEVAL PERIOD (A.D. 1200 to A D 1800)

In this period, very little original work was done There were foreign invasions in North India; therefore mathematical development was checked in this period. At this time, South India was considered as the centre of learning science and mathematics. During this period, Kamlakar (A.D. 1600) wrote *Siddhanta-Tatva.* Vivek Narain wrote *Ganit-Kaumudy* and Neel Kantha (in A.D. 1587) wrote *Tajik Neel Kanthia.* Pandit Jagannath (A.D. 1731) translated the works of Ptolemy and Euclid into Sanskrit. This book is known as *Samrat-Siddhanta.* The present terminology of geometry is mostly taken from this book.

Those who wish to know more about the Indian contributions to mathematics in early periods may consult the following source books :

1. *The History of Hindu Mathematics* by A. D. Singh and V. P. Dutta.
2. *The History of Ancient Indian Mathematics* by C. N. Srinivasaiyengar (World Press Pvt. Ltd., Calcutta, 1967).

3. *Prachin Bharti Ganit* by Dr. B.L. Upadhya (Vigyan Bharti, New Delhi).
4. *Ganit Ka Itihas* by Dr Brij Mohan (Kashi Vishvavidyalaya, U.P.).

**Development of Mathematics in Greece—A Shift in Emphasis**

The Greeks transformed mathematics into something vastly different from a set of empirical conclusions worked out by their predecessors. The Greeks insisted that mathematical facts must be established not by empirical procedures, but by deductive reasoning (i.e., reaching conclusions on the basis of a set of certain statements called hypotheses). In philosophical speculations, reasoning centres on abstract concepts and broad generalisations, and is concerned with truths and with inevitable conclusions following from (assumed) premises It is deductive reasoning that philosophers find to be their indispensable tool, and so the Greeks naturally gave preference to this method when they began to consider mathematics. The significant thing from the point of view of our present study is that the ancient Greeks found in deductive reasoning the vital element of the modern mathematical method.

**The Mathematical Method**

Earlier, the Greek mathematicians and philosophers held the truth of material as self-evident. The fundamental assumptions, known as postulates or axioms of Euclid, were accepted as 'obviously true'. After some twenty centuries (about the middle of the 19th century) came a revolution in mathematical thought Lobachevski and Bolyai announced the discovery of non-Euclidean geometries, shaking the beliefs of centuries to the roots. The facts of mathematics were no longer self-evident (i.e., absolute) truths—all mathematical assertions were regarded henceforth as relative and contingent truths. More clearly a mathematical statement is to be taken as valid or not valid with reference to **certain mathematical (axiom) systems.** The same mathematical statement can be valid in one system and not valid in another system. For example, in plane geometry the statement "the three angles of a triangle are together equal to 180°" is a valid theorem but in spherical geometry the same statement does not hold. We should not be surprised if a mathematical statement holds in a system and does not hold in another. In fact, we have no dearth of examples in real life situations to illustrate the status of mathematical statements. Consider the statement "Death penalty is the maximum punishment which can be given to a criminal under law". This statement is valid in India, while invalid in England. The reason is that while in India certain crimes are punishable with death penalty, in English law there is no death penalty.

Similarly, in mathematics we have statements (e.g., identities or inequality) which hold under certain conditions, but are false under certain other conditions. For example, the statement "all angles of the triangle are equal" holds for equilateral triangles, but is false for all other triangles.

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Q.1. Identify two statements from real life situations which are true in one context and false in another.

Q.2. Give two mathematical statements which are true under certain conditions, but false under certain other conditions.

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## Deduction and Induction

Thus, mathematics consists of the study of different parts which are in themselves different mathematical structures. Before we proceed to discuss the nature of mathematical structures, it will not be out of place to emhasise the special nature of the mathematical method as distinct from the method of science. The mathematical method tells us under what circumstances a mathematical statement is held valid in a mathematical system.

The method by which we come to a conclusion in science is called inductive reasoning. Inductive reasoning is reaching a conclusion by [illegible] [illegible]enon. For example, [illegible] by actual measurement that the sum [illegible] to 180°. After measuring the angles of several triangles, we reach the conclusion **inductively** (or by induction) that the sum of the angles of a triangle is 180°.

To find out the maximum number of segments in which a circle can be divided by joining *n* points on its circumference, a student used inductive reasoning as below :

| *No. of points on* [illegible]*ce 'p'* | *Maximum no. of segments 'n'* |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 4 |
| 4 | 8 |

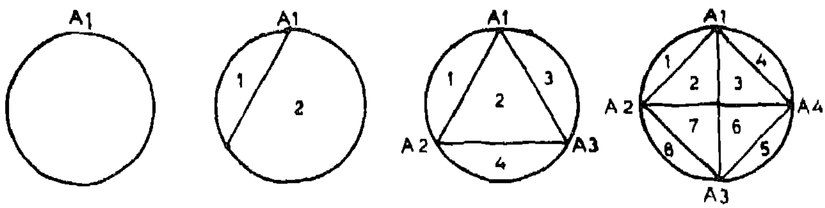


Fig 1 1

He got the following formula

$$p = 2^{n-1}$$

(i) Verify, whether it is true for $n = 5$ ?

(ii) Can we say, generalizations established on the basis of inductive reasoning are always true ?

Q. What are the limitations of generalizations based on inductive reasoning ?

Deductive reasoning is reaching a conclusion on the basis of a set of certain statements called *hypotheses* Rules of logic (deductive logic) are applied in deduction. The acceptance of the conclusion depends upon the acceptance of the hypothesis. If we take the hypothesis for granted, we are **forced** to take the conclusion as granted in a **valid** deduction. If there is a reasoning by which we take the hypothesis for granted but not the conclusion, the reasoning is considered invalid. The hypothesis must imply the conclusion under all circumstances in a valid deduction The customary Q.E.D. 'quad erat demonstrandum' or 'which was to be demonstrated' simply means that the theorem proved is a conclusion of a valid deduction from the hypothesis. In what follows we will try to describe mathematics as a system wherein deductive reasoning is used.

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Q.1 Suppose in a mathematical system $P$ and $Q$ are two true statements . If a theorem states that "if $P$, then not—$Q$", is the deductive reasoning used in the theorem valid ? Support your answer.

Q.2. Why can we **not** prove in **plane** geometry that the sum of three angles of a triangle is together equal to 270° ?

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**Mathematical System**

As we have remarked earlier, mathematics includes many components

which are in themselves mathematical structures or mathematical systems. A typical mathematical system has the following four parts :

(1) Undefined terms
(2) Defined terms
(3) Axioms, and
(4) Theorems.

*Undefined Terms* : In geometry or in any other mathematical system, we have to start with some terms which are taken as undefined terms and other terms of the system are defined in terms of undefined terms. If we try to define all the terms of a system, we will end in making cyclic definition. That is why we have to select some terms as undefined terms

For example, suppose we define point as 'the common point of two intersecting lines' and a line as 'the path described by a point in moving along the path of the shortest distance between two points' Then the point and the line are defined in terms of each other and the definitions are faulty due to being circular.

So, we have to select a few of the terms as undefined so that the terms are defined in terms of these undefined terms The choice of the undefined terms is completely arbitrary and generally to facilitate the development of the structure.

*Defined Terms* . After fixing undefined terms we should try to define other terms of the system in terms of undefined terms, using in these definitions the common articles and the connectives of languages. A good definition should satisfy the following conditions :

(1) A definition should be consistent, i.e., the definition should convey the same meaning of the term in all possible situations of the system.
(2) A definition should consist of only undefined terms or other previously defined terms, besides the common articles and connectives. A definition using the word 'curved' will not be meaningful in the system if the term 'curved' is not an undefined term or not defined earlier.
(3) A definition should be stated clearly and precisely without redundancy. In the definition of an equilateral triangle as 'a triangle having three equal sides and three equal angles' either the phrase 'three equal sides' or the phrase 'three equal angles' can be deleted as one of these is redundant, if the other is to be kept in definition.

*Axioms* . Axioms or postulates are statements in a mathematical system which we take for granted and these statements (axioms) describe the relationships existing among the undefined terms of the system.

For example, in plane geometry, the axioms (1) through a point external to a given line, one and only one line can be drawn parallel to the

given line, and (2) 'two lines meet at a point' describe the relationships existing among the undefined terms 'points' and 'lines'.

*Theorems* · In ordinary life, we use generally a form of argument called the rule of implication. The rule of implication states that if (1) the statement $p$ implies the statement $q$, and (2) the statement $p$ is true, then the statement $q$ will be true.

When we apply the rule of implication to the axioms, we generate new statements and again we may apply this rule to these new statements.

A statement to which we arrive at by successive application of the rule of implication to the axioms and statements previously arrived at is called a theorem. The sequence of steps through which we arrive at the theorem is called the **proof** of the theorem. As an illustration, we give the following proof of the theorem · "The sides opposite to equal angles in a triangle are equal".

Given : A triangle $ABC$ in which $\angle B = \angle C$.

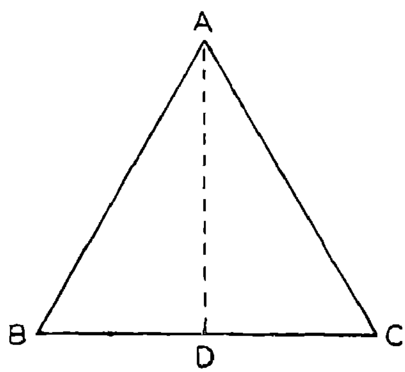



Fig. 1.2

To prove $AB = AC$

Construction · Draw the bisector of $\angle BAC$ to meet $BC$ at $D$.

*Proof* : In triangles $ABD$ and $ACD$,

$$\angle ABC = \angle ACB \text{ (Given)}$$
$$\angle BAD = \angle CAD \text{ (by Construction)}$$
$$AD = AD$$

Therefore $\triangle ABD \equiv \triangle ACD$ (rule of implication applied to two triangles $ABD$ and $ACD$)

Hence $AB = AC$ (Rule of implication applied to $\triangle ABC$)

Q.1. Select some examples of bad definitions in mathematics and explain why they are bad.

Q.2. Explain *how* the rule of implication is used twice in the proof of the theorem given above as illustration.

Q.3. Select the proof of a theorem from a geometry textbook and explain how the rule of implication is used in the proof.

## Types of Proof in Mathematics

(*a*) *Direct Proof*: If you recall your school geometry, you must be remembering that you proved many theorems. In every theorem you had certain statements as hypothesis and certain statements to be proved or deduced from the hypothesis. After stating **directly** what is to be proved (conclusion), you start to write down the proof. While writing the proof you write a sequence of statements, each statement being justified by an axiom or a previous theorem or a definition. The proof ends with the statement of the conclusion. This method of writing the proof is called **direct proof.** The illustrative proof of the theorem "The sides opposite to equal angles of a triangle are equal" in this chapter is an example of a direct proof.

(*b*) *Proof by Mathematical Induction* : Though this method of proving a theorem is called mathematical induction, it is not an inductive method. It is deductive in nature. This method depends upon the following principle . If (*a*) $P(n)$ is a statement involving the positive integral variable $n$, (*b*) $P(n)$ holds for $n = 1$, i.e, $P(1)$ is true, and (*c*) for every positive integral $n$, the truth of $P(n)$ implies the truth of $P(n+1)$, then $P(n)$ holds for all positive integral values of $n$. This method is used in mathematics. As an illustration, let us apply mathematical induction in proving this statement :

$$1+2+3+4+ \dots .+n = \frac{n(n+1)}{2}$$

Let $P(n)$ denote the statement to be proved. Note that $P(n)$ is meaningful for positive integral values of $n$. So the condition (*a*) is satisfied.

$P(1)$ turns out to be "$1 = \frac{1(1+1)}{2}$" which is true, so the condition (*b*) is satisfied.

Now suppose that $P(n)$ is true

$$\text{i.e.,} \quad 1+2+3+\dots \dots+n = \frac{n\,(n+1)}{2}$$

Adding (n+1) to both sides we have,

$$1+2+3+\ \ldots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$$

i e.,
$$1+2+3+\ldots+n+(n+1)=(n+1)\left(\frac{n}{2}+1\right)$$

i e.,
$$1+2+3+\ldots n+(n+1) = \frac{(n+1)\ (n+2)}{2}$$

$$\text{i.e. } 1+2+3\ldots n+(n+1) = \frac{(n+1)\ (n+2)}{2}$$

Now on substituting $n+1$ for $n$ in the statement, we get

i.e ,
$$1+2+3+\ldots+(n+1) = \frac{(n+1)\ [(n+1)\ +1]}{2}$$

i.e., $P(n+1)$ is true

So we have proved that if $P(n)$ is true, then $P(n+1)$ is true Thus the condition (*c*) is also satisfied. Hence by applying mathematical induction we prove that $P(n)$ is true for all positive integral values of $n$.

(*c*) *Proof by Contradiction*· An important method of proving a theorem is proof by contradiction. This method is based on the following principle: For every statement $X$, either $X$ or its negation $\bar{X}$ is true, but *not both*. So if we are able to prove that $\bar{X}$ the negation of a statement $X$ is false, then we have proved the truth of the statement $X$.

As an illustration suppose we wish to prove that "in a triangle the side opposite to the greater angle will be greater than the side opposite to the smaller angle". To start with, let us have $\triangle ABC$ in which $\angle ABC >$ $\angle ACB$.......(1)

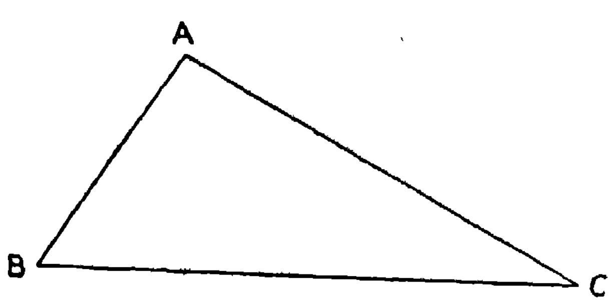



Fig. 1.3

Then we will suppose that the conclusion of the theorem does not hold i.e., $AC \ngtr AB$, and arrive at a contradiction by using (1) This indicates

that $AC \ngtr AB$ does not hold under (1) i.e., the statement $AC > AB$ is true under condition(1).

(*d*) *Proof of the Contrapositive* · In proof of the contrapositive method, instead of proving the theorem in the form "If $p$, then $q$", we prove the theorem in the form "If not-$q$, then not-$p$" As an illustration, let us take the theorem again : "In a triangle $ABC$, if $A > B$, then $a > b$." Here let us suppose $p$ means "$A > B$" and $q$ means "$a > b$" Then in direct proof we have to prove that "If $A > B$, then $a > b$", i e., "if $p$, then $q$".

But in proof of the contrapositive, we have to prove that "if not-$q$, then not-$p$". Here not-$q$ means "$a \ngtr b$" and not-$p$ means "$A \ngtr B$". So in the proof of the contrapositive we will prove that "if $a \ngtr b$, then $A \ngtr B$."

It should be noted that in direct proof and in proof by mathematical induction we do not consider the negation of a statement But in the proof by contradiction (which is also called indirect proof) and proof of contrapositive we consider the negation of a statement for the proof.

(*e*) *Disproof of a Statement by a Counter Example* · In mathematics, for proving a statement we have to give a proof, but a single instance wherein the statement does not hold will disprove the statement. Such an instance is called a counter example

For illustration to disprove the statement "$2^n - 1$ is a prime" one counter example, i e., $2^4 - 1 = 15 = 3 \times 5$ is sufficient.

Another good illustration of this is as follows :

To disprove that "Two figures having congruent angles are similar" we can cite a set of a square and a rectangle as a counter example to this statement.

---

Q.1. Give two illustrations of each of the following from school mathematics :

(*a*) Direct proof
(*b*) Proof by contradiction
(*c*) Proof of the contrapositive, and
(*d*) Counter example.

Q.2. Give an illustration of a proof by mathematical induction.

Q.3. Why is the method of mathematical induction *not* of much use in geometry ?

---

(*f*) *Synthetic Proof* : The usual procedure in writing down a proof of a theorem is that we start from the hypothesis and arrive at the conclusion of the theorem. This procedure of writing down the proof is known

as synthetic method or synthetic process of proving a theorem. Th illustration of proof given on page 12 is an example of synthetic proof.

---

Q What is the difference between 'direct proof,' and 'synthetic proof'

---

(g) *Analytic Proof*. But many times when the statement to be provec is complicated, it is not easy to write the synthetic proof of it. Then w try to analyse the statement, starting from the conclusion and move back wards step by step, finally reaching the hypothesis. This way of arguin backwards (from conclusion to hypothesis is known as analytic method) In fact, most of the proof is first constructed by applying analytic methoc and then written in a finished form using synthetic method. As ai illustration of analytic method, we give the proof of the following algebrai assertion : If $a+b+c = 0$, then $a^3+b^3+c^3-3abc = 0$

*Proof*. $a^3+b^3+c^3-3abc = 0$

if $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$ is true which is true

if $a+b+c = 0$ which is the hypothesis of our assertion. So th assertion is proved.

---

Q.1. Give one example of analytic proof in geometry.

Q.2. Give two examples of synthetic proof from trigonometry.

Q.3. Give two examples of analytic proof from trigonometry.

Q.4. As soon as we have an analytic proof of a theorem, we also hav a synthetic proof of the same. How ?

---

**Language in Teaching Mathematics**

In teaching mathematics, the teacher uses ordinary language t communicate mathematical concepts Ordinary language being vague ambiguous and emotive, the mathematics teacher in a classroom has t combat with the resulting hazards. Let us analyse and see what is at stak while one uses ordinary language to teach mathematics in a classroom The general aims of teaching mathematics being development of logica thinking, reasoning, precision and objectivity, there is the danger of crea ting misundersanding rather than facilitating understanding while one use ordinary language. The danger is all the more great, when the languag

is loose and used indiscreetly. We shall have illustrations of this in due course.

In mathematics, definitions and conventions play a decisive role. The results derived depend on the choice of definitions To cite an instance, consider the two definitions of parallel lines given below :

$D_1$ · Two lines $a$ and $b$ are parallel if $a$ and $b$ have no common point.

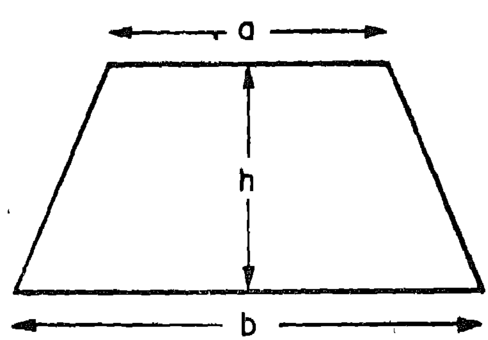
$D_2$ · Two lines $a$ and $b$ are parallel if $a$ and $b$ do not intersect.

Further, two lines are said to intersect, if they have one and only one common point.

According to $D_1$ coincident lines are *not* parallel while according to $D_2$, coincident lines are also parallel. Another instance is the definition of a trapezoid for which we give two definitions below :

$D_1$ · A quadrilateral (in a plane) is said to be a trapezoid (or trapezium) if one and only one pair of opposite sides is parallel.

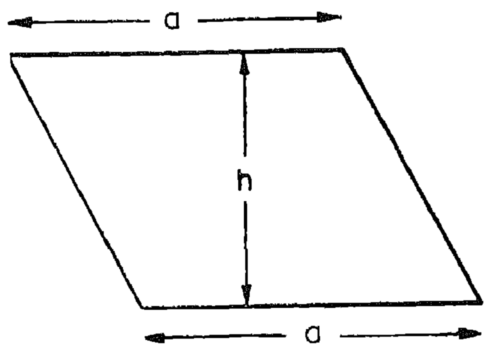
$D_2$ A quadrilateral (in a plane) is said to be a trapezoid if at least one pair of opposite sides is parallel. If we accept $D_1$, a parallelogram is not a trapezoid while according to $D_2$ a parallelogram is also a trapezoid. However, it is significant to note that some results of trapezoid hold in the case of a parallelogram as well. One such result is the area of a trapezoid = $\frac{1}{2}$ (sum of parallel sides) $\times$ height.




$$A = \frac{1}{2}(a+b)\,h$$

Fig. 1.4

Now we illustrate the remark we made earlier, namely, the ordinary language one uses creates misunderstanding and confusion. For instance, in a problem on permutation and combination or probability we come across the statement—'a box contains red and blue pencils from which one has to draw'. This statement may mean either (1) pencils are double-




$$A = \frac{1}{2}(a+a)\,h = ah$$

Fig. 1.5

coloured, viz., both red and blue or (2) some pencils are red while others are blue.

Furthermore, we use such imprecise language as in—"one triangle is placed on another in such a way that a side of one falls along a side of the other, a vertex of one falls on a vertex of the other and so on," as if the triangles (about which mathematical relations are being established) are material objects which can be physically handled. On the contrary, when proofs of congruency of triangles are sought in this way, the principles of superposition involving rigid motions are being used It is to be understood that material objects resembling mathematical objects are at best analogies and not the mathematical objects themselves. Thus while teaching the notion of a triangle and the properties thereon, a hard-board piece in the shape of a triangle may serve the teacher as an aid provided the teacher knows when to cast off the material piece.

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Q. Observe your friend teaching mathematics to a class. Isolate instances of the language used by him which create confusion or difficulties for the students.

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**Problem Solving**

Solving a problem means to find a way out of difficulty, a way to remove an obstacle, and it is a characteristically human activity even though animals solve their problems too. While there is no royal road to success

in solving any given problem, one can think of a blueprint which works in problem solving. 'If you wish to learn swimming, you have to get into water', says an old proverb and the same holds good for problem solving Our knowledge about any subject consists of information and of know-how. In mathematics, know-how is the ability to solve problems—not merely the routine ones, but problems requiring some degree of independence, judgement, originality and creativity. Therefore, the emphasis on methodical work in problem solving is first and foremost. It is very essential to lay stress on problem solving since the ability to solve a problem is a reliable evidence of one's assimilation of the particular theory used in the problem ; besides it encourages creative thinking.

G. Polya offers a recipe for problem solving in his book *How to Solve it* ?

**First** : Understand the problem

This means :

1. What is the unknown ? What are the data ? What are the conditions ?
2. Is it possible to satisfy the condition ? Is the condition sufficient to determine the unknown or is it insufficient ?
3. Draw a figure , introduce suitable notation.
4. Separate the various parts of the condition (s). Can you write them down ?

**Second** : Devise a plan

This means .

1. Have you seen it before ? Or have you seen the same problem in a slightly different form ?
2. Do you know a related problem ? Do you know a theorem that could be used ?
3. Look at the unknown! And try to think of a familiar problem having the same or similar unknown.
4. Here is a problem related to you and solved before. Could you use it ? Could you use its results ? Could you use its methods ? Should you introduce some auxiliary element in order to make its use possible ?
5. Could you restate the problem ? Could you restate the problem differently ? Go back to definitions.
6. If you cannot solve the proposed problem, try to solve first some related problem Could you imagine a more accessible related problem ? A more general problem ? An analogous problem ? Could you solve part of the problem ? Keep only a part of the condition, drop the other part, how far is the unknown determined ? How can it vary ? Could you derive something useful from the data ? Could you think of other data appropriate to

determine the unknown ? Could you change the unknown or the data or both, if necessary, so that the new unknown and the new data are nearer to each other.

**Third** : Carry out the plan

This means carrying out the plan of the solution Check each step Can you see clearly the step is correct ? Can you prove that it is correct ?

**Fourth** : Examine the solution obtained

This means :

1. Can you check the results ? Can you check the argument ?
2. Can you derive the result differently ? Can you see it at a glance ?
3. Can you use the result or the method, for some other problem ?

ILLUSTRATION

**Problem** . Given a right angled triangle $ABC$, construct a point $N$ inside the $\triangle$ $ABC$ such that the angles $NBC$, $NCA$ and $NAB$ are equal.

Now let us identify the different stages of problem-solving in the solution of this problem.

**Step 1** : Understanding the Problem

Since to start with this construction is not an easy one, an analysis of the problem is necessary for understanding the problem. For this, denote the angles $BAC$, $ABC$, by $\alpha$, $\beta$ respectively and assume $\angle ACB = 90°$. It

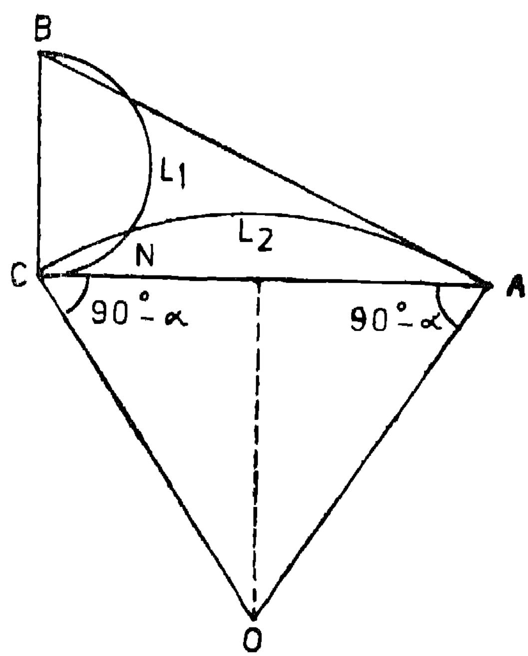


Fig. 1 6

is better to draw the figure for the clarification of the import of tl problem.

Now $N$ has to be a point inside $\triangle ABC$ such that

$$\angle NBC = \angle NCA = \angle NAB.$$

So

$$\begin{aligned}\angle BNC &= 180^\circ - (\angle BCN + \angle NBC)\\ &= 180^\circ - (\angle BCN + \angle NCA)\\ &= 180^\circ - 90^\circ\\ &= 90^\circ\end{aligned}$$

Similarly we have

$$\angle ANC = 180^\circ - \alpha,\ \angle ANB = 180^\circ - \beta$$

**Step 2 :** Devise a plan

After analysis of the problem in step 1, we have the following importar information :

$$\angle BNC = 90^\circ$$

$$\angle ANC = 180^\circ - \alpha,\ \angle ANB = 180^\circ - \beta$$

This information helps us to plan how to construct the point $N$. Sinc $\angle BNC = 90^\circ$, the point $N$ will lie on the semi-circle with $BC$ as th diameter and within $\triangle ABC$.

Also since $\angle ANC = 180^\circ - \alpha$, the point $N$ will also lie on the minoı arc of a circle which subtends at its centre $O$ an angle of $2\alpha$.

Now we can draw a plan for construction .

1. Draw a circle $L_1$ with $BC$ as the diameter.
2. Draw a ray from $A$ which is perpendicular on $AB$ and makes ar angle $90^\circ - \alpha$ with $AC$.

Draw another ray from $C$ which makes with $CA$ an angle $90 - \alpha$.

Let these rays from $A$ and $C$ intersect at $O$ Draw the minor arc $L_1$ of chord $AC$ with $O$ as centre and $OC$ as radius. Let this minor arc $L_1$ intersect the circle $L_1$ at the point $N$ which is the required point.

**Step 3 :** Carrying out the plan

In this problem, carrying out the plan involves actually drawing the construction Carrying out the plan is essential because then only the defects or any drawbacks of the plan will come into limelight. Foı example, in this problem, we should be sure that our plan of construction is such that the point $N$ should be *within* $\triangle ABC$.

Generally in theoretical problems, carrying out the plan requires that we do the actual calculations, etc. Here we have to draw the construction and prove (to ensure) that the point $N$ lies inside the triangle $ABC$.

**Step 4 :** Examining the solution obtained and looking back

In the final step, we should evaluate our solution and examine its different ramifications.

After analysis of the problem, we have the following data .

$\angle BNC = 90°$, $\angle ANC = 180°-\alpha$, $\angle ANB = 180-\beta$

Of these, we have not used the last data, *i.e.*, $ANB = 180°-\beta$ in the plan for construction. But this datum can be used for the verification of our plan. For example ;

$\angle BNC + \angle ANC + \angle ANB$

$= 90° + 180° - \alpha + 180 - \beta$

$= 450° - (\alpha + \beta)$

$= 450° - 90°$ (because $\alpha + \beta = 90°$)

$= 360°$

This verification gives additional evidence to our method of construction.

Also, our method of construction gives clues to the solution of the problem.

**Problem** : Construct a point $N$ inside an arbitrary $\triangle ABC$ such that $\angle NBC = \angle NCA = \angle NAB$.

---

Identify the stages of problem-solving in the solution of the following problems and solve them :

1. Construct a triangle $ABC$, when $D, E, F$, the feet of altitudes, are given. Express the sides of $\triangle ABC$ in terms of those of $\triangle DEF$.

2. The area $T$ and angle $C$ of a triangle $ABC$ are given. Find the lengths of the sides $a$ and $b$ so that the side $c$ is as short as possible.

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CHAPTER 2

# Commercial Mathematics

## Introduction

We come across many situations relating to money transactions. Such transactions may be in the form of money or material. Mathematical processes are involved in these transactions Look at the following situation ·

You bought goods worth Rs. 2.85 p and gave Rs. 10.00 to the shopkeeper. Sometimes the shopkeeper returns Rs 8 15 p instead of Rs. 7.15 p and thus bears a loss of Re. 1. The reason is self-evident. The shopkeeper subtracts 0.85 p from Re. 1 and Rs. 2 from Rs. 10.00. He forgets to carry over Rs. 1. To avoid this blunder the shopkeeper usually first pays paise 15 and counts Rs. 3. After this he counts Rs. 4 and Rs. 5 and completes Rs. 10.

In this situation subtraction is a mathematical process applied to a daily-life problem.

The notion of "at the rate of" is given considerable importance in business procedures such as discount, stocks and bonds, saving account, simple and compound interest, government taxes, instalment buying, wholesale and retail trade, etc Percentage is a convenient and useful mathematical process to express the idea of "at the rate of". Topics like profit and loss, commission and discount, shares and dividends, simple and compound interests have been included in Business Procedures as applications of percentage. In this unit, we shall discuss how we can explain to the students the application of percentage in profit and loss, commission and discount, government taxes, simple and compound interest, trade discount and in instalment buying.

## Content to be Covered

(1) Revision of :

(A) Meaning of per cent.

(B) To calculate per cent of the given amount.

(C) Following types of problems relating to (A) and (B) ·

1. To find the per cent of a number.

2. To find what per cent of one number is another.
3. To find the number when a specified percentage of it is given.

(D) Problems of daily life involving the use of percentage ·

1. To calculate profit and loss.
2. To calculate commission and discount
3. To calculate dividends and shares.
4. To calculate taxes
5. To calculate simple interest.
6. To calculate compound interest.
7. To calculate trade discount
8. To buy in instalment

**Revision from A to C**

Out of the above units, sub-units ($A$) and ($B$) are taught in Classes VI to VIII. Students did a lot of practice to learn ($A$) and ($B$). Under $C$ (1), it is important to teach that "100 per cent of any number equals the number". For example, 100 per cent of 20 is 20 and 100 per cent of 1334 is 1334. Three types of problems under ($C$) should be comprehended by the students Often students try to search out clues to classify the problems of $C$ (1), $C$ (2), and $C$ (3) and solve them by rote. This defeats the purpose of meaningful learning. Classify problems into $C$ (1), $C$ (2), $C$ (3)—using clues reduces the reasoning process to a minimum which is undesirable.

**Revision of C**

We know that 25 per cent of 60 is equal to 15 Here 25 is called the rate, 60 is called the base and 15 is called the percentage.

If any two quantities are known, the third can be calculated. Problems under ($C$) can be solved by using this formula.

Problems under ($D$) have been included in Classes IX-X under unit of Commercial Mathematics.

Rate, base and percentage are related as below ·

$$\frac{\text{Rate} \times \text{Base}}{100} = \text{percentage}$$

**Profit and Loss**

Students have solved direct and indirect problems on profit and loss in previous classes. Here we will include those problems which are a bit complicated. The main purpose of including such problems is to inculcate problem solving ability among the students.

Look at the following problem .

**Problem 1**

The price of brinjals is 25 per cent more than that of pumpkin. By what percentage is the price of pumpkin less than that of the brinjals ?

**Solution**

Now let us analyse the problem

*(1) Understanding the Problem*

Price of the brinjals is 25% more than that of the pumpkin. It means

$$25 = \frac{\text{Difference in price of the brinjals and pumpkin}}{\text{Price of pumpkin}} \times 100.$$

Similarly

Price of the pumpkin will be less than that of brinjals in percentage

$$= \frac{\text{Difference in price of the brinjals and pumpkin}}{\text{Price of brinjals}} \times 100.$$

*(2) Devising a Plan*

Assume the price of 1 kg of pumpkin to be 100 paise and calculate the price of brinjals. Calculate less percentage of the price of pumpkin from the price of brinjals

$$\text{Less } \% = \frac{\text{Difference in Price}}{\text{Price of Brinjals}} \times 100$$

*(3) Carrying out the Plan*

Let the price of one kg pumpkin be Re 1 or 100 p
Price of one kg. brinjals $= 100 + 25\%$ of 100 p
$= 125$ p

Price of one kg. of pumpkin is 25 p less than one kg. of brinjals.

$$\% \text{ less } = \frac{25 \times 100}{125} \%$$

$$= 20\%$$

**Problem 2**

If the price of cooking gas is increased by 10%, then by what per cent should its consumption be reduced so as not to increase the expenditure ?

**Solution**

(*1*) *Understanding the Problem*

After the increase of the price of gas, less gas will be available for the same amount. Therefore, consumption will have to be reduced so as not to increase the expenditure.

(*2*) *Devising a Plan*

Assume that the expenditure on gas is Rs. 100 and work out the problem.

(*3*) *Carrying out the Plan*

Let the expenditure on gas be Rs. 100.00

On increasing the price, same quantity is available in Rs. 110.

$\therefore$ Quantity of gas available for Rs. 10 should not be consumed to keep the expenditure same

$$\therefore \text{ \% reduction in consumption} = \frac{10}{110} \times 100\%$$

$$= 9\frac{1}{11}\%$$

---

Q.1. Prepare a plan to acquaint the students with the steps of problem-solving using the following problem :

A bicycle is sold at 20% profit. If it had been sold at 20% less, the selling price would have been Rs. 100 less. Find the cost of the bicycle

Q.2. Devise a plan and carry out the same to solve the following problem :

An article costing Rs 'C' is sold for Rs. 100 at a loss of X per cent of the selling price. It is then resold at a profit of X% of the new selling price Rs. 'S'. If the difference between S and C is Rs. $1\frac{1}{9}$, then find the value of X.

Q.3. Devise a plan to solve the following problem :

(*a*) Problem 1 : The price of an article is cut by 10%. To restore it to its former value, by what % the new price must be increased.

(*b*) Problem 2 . A merchant placed on display some dresses, each

with a marked price. He then posted a sign "1/3 off" on those dresses. The cost of dresses was 3/4 of the price at which he actually sold them. Find the ratio of the cost to the marked price.

---

**Commission**

You know that an agent or a salesman often sells goods on commission basis. This commission is often expressed as a rate in percentage. For example, an insurance agent may receive 10 per cent of the premium deposited on the policy made by him. If the premium deposited is Rs. 18,000 in a year, he receives 10 per cent of 18,000.00 $= \frac{10}{100} \times 18{,}000 = 1{,}800.00$ as his commission The rate is 10 per cent There are three types of problems in commission :

(1) Given the commission rate and the value of goods, to find the salesman's commission

(2) Given the value of the goods sold and the amount of salesman's commission, to find the commission rate in per cent.

(3) Given the salesman's commission rate in per cent and the amount of his commission, to find the cost of the goods.

The students should classify the problems under C(1), C(2) or C(3). Let us study this by taking a few problems :

**Problem 1**

An agent's commission rate is 10 per cent. He received orders worth Rs. 10,000 in July, 1982. How much did he get as commission ?

**Solution**

Let us find out what is given. You will see that in this problem rate and value of goods sold are given. What is to be calculated ? Commission. Now you can devise the plan to solve the problem. Here the commission rate and the value of the goods ordered are related as given below :

In this problem, by the formula .

$$\begin{aligned} \text{Commission} &= \text{rate} \times \text{value of goods ordered.} \\ &= 10\ \%\ \text{of}\ 10{,}000 \\ &= \text{Rs. } 1{,}000.00 \end{aligned}$$

**Problem 2**

A salesman sells goods worth Rs. 20,000. He receives Rs. 5,000 as commission. What is the commission rate ?

**Solution**

*Note* : In this problem value of goods ordered (base) and commission (percentage) are given and rate is required ·

$$\frac{\text{Rate} \times \text{Base}}{100} = \text{Percentage}$$

$$\text{Rate} = \frac{\text{Commission} \times 100}{\text{value of goods sold}}\%$$

$$= \frac{5{,}000 \times 100}{20{,}000}\%$$

$$= 25\%$$

**Problem 3**

A salesman's commission rate is 10%. He receives commission amounting to Rs. 1,500 in January. How much goods did he sell in January ?

**Solution**

*Note* : In this case the value of goods sold is required.

10% of the value of goods sold = Rs. 1,500.00

$$\frac{10}{100} \times \text{value of goods sold} = 1{,}500.00$$

Value of goods sold = 1,500 × 10 = Rs. 15,000.00

The above three problems are solved by the direct use of formula. But sometimes the problem cannot be classified as a single case. Let us consider the following problem .

**Problem 4**

A television salesman receives 5% commission and a flat salary of Rs. 500 per month. How many television sets must he sell per year at Rs. 5,000 each, to make an average salary of Rs. 1,500 per month ?

*Note* : This problem can be split up into three sub-problems

(*i*) Amount of commission required by selling television.

(*ii*) Number of sets required if percentage rate and commission are given.

(*iii*) To calculate the cost price of a television set.

Question arises which of the sub-problems could be solved first ·

**Solution**

Commission on one set $= 5\%$ of Rs 5,000 $= \frac{5}{100} \times 5{,}000 =$ Rs. 250.

His flat salary is Rs. 500 per month

He wishes to earn Rs. 1,500 per month.

He must earn (Rs. 1,500—500), i.e , Rs. 1,000 as commission per month.

So his average commission per year = Rs. $1000 \times 12$ = Rs. 12,000

So the number of television sets he must sell per year

$$= \frac{12{,}000}{250} = 48$$

This problem can be solved in any one of the several slightly different ways. There are two reasons for considering problems such as above , one reason is that they are related to real life situation and secondly, they illustrate the fact that not all problems involving commission fall directly under C(1), C(2) and C(3).

**Discount**

The problems on discount are often classified into three cases which are similar to the problems on commission. Besides, sometimes two (or more) successive discounts are allowed. The first discount may be one offered to all customers, and the second discount may be for cash payment. Newspapers and magazines often carry advertisement on discount sale.

**Example**

A radio-set is sold for Rs. 500 less successive discounts of 10% and 5%. Find the discount price.

**Solution**

Here list price of the article and two successive discounts are given. After calculating the first discount, the second discount shall be calculated on the balance. Similarly, the third discount (if any) is to be calculated on the balance after subtracting the second discount from the first balance.

The selling price of the article will be the last balance.

First Discount $= 10\%$ of Rs. 500 $= \frac{10}{100} \times 500 =$ Rs. 50

Second Discount will be allowed on Rs. 500 —50, i.e., Rs. 450

$$= 5\% \text{ of Rs. } 450 = \frac{5}{100} \times 450 = \text{Rs. } 22.50$$

Discount price = 450.00—22.50 = Rs. 427 50

*Note* : The student must understand that it is better for the purchaser to receive discount of 15% than to have successive discounts of 10% and 5%.

**Example**

Find the single discount equivalent to two successive discounts of 10% followed by 5%.

**Solution**

Let the list price of an article be Rs. 100

Balance after first discount = 100—10 = Rs. 90.00

Second discount = 5% of Rs. 90 = $\frac{5}{100} \times 90$ = Rs. 4.50

Balance after second discount = 90.00—4.50 = Rs 85.50

Hence the single discount equivalent to the successive two discounts is the sum of the two discounts calculated above

Equivalent single discount = 10+4.50 = 14.5%

The teacher should take up the problems on successive discounts after solving some problems under three simple cases C(1), C(2) and C(3). He should also take example from Khadi shops where successive discounts are allowed.

A problem on commission and discount can be split up into several simple problems. (The order of solving these simple problems in succession depends upon the solution of the main problem.)

**Problem**

A merchant buys goods at 25% off the list price. He desires to mark the goods so that he can give discount of 25% on the marked price and still clear a profit of 25% on the selling price. What per cent of the list price must he mark the goods ?

This problem can be split up into following sub-problems :

(*a*) If the list price of an article is Rs. 100 and it is sold to a merchant at 25% off the list price, then how much money has the merchant to give to purchase it ?

(*b*) If the merchant marks the goods at Rs. $x$, what amount of discount will he give if it is 25% of marked price ?

(*c*) What will be the selling price of the goods in terms of $x$ ?

(*d*) If the merchant gets 25% profit of selling price, what will be the price of the goods which the merchant pays ?

(*e*) Compare the prices in (*d*) and (*a*) and find the value of $x$.

(*f*) Will the value of $x$ be the per cent of the list price that the merchant has to mark the goods ?

---

Q.1 Solve the main problem by solving problems from (*a*) to (*f*).

Q 2 You know the steps involved in problem solving .

In which step will you include the splitting of the problems into sub-problems ?

---

**Cash Discount**

You have already learnt about successive discount. Successive discount is given by a wholesale dealer to the retailer at different rates, depending on the time taken in making the payment. For example, terms of payment may be :

| | |
|---|---|
| For immediate payment | 4% off |
| For payment within 10 days | 3% off |
| For payment after 10 days but before 20 days. | 1% off |
| After 20 days but within 30 days | Net due of the bill |

Terms of payment are symbolically expressed as :

Terms : 4, 3/10, 1/20, $n$/30

Here successive discount is paid for cash payment and hence called Cash Discount.

Look at the invoice of the goods purchased by a retailer from a wholesale dealer

INVOICE

| *Quantity* | *Particulars* | *Rate* | *Total* |
|---|---|---|---|
| 12 Dozen | Shoe No. 7 Bumpy Brand | Rs. 80 per dozen | Rs. 960.00 |
| | Less : 20% Discount | | Rs. 192.00 |
| | | Net amount | Rs. 768.00 |

$$\text{Terms : } 4, \frac{3}{10}, \frac{1}{20} \text{ and } \frac{n}{30}$$

Here net amount payable is Rs. 768 00. Term 4 means 4% of immediate cash payment, term 3/10 means 3% discount if paid within ten days, term 1/20 means 1% off if paid after ten days but before 20 days, $\frac{n}{30}$ means net due.

If the retailer makes payment on the 7th day, he will get 3% off Therefore successive discount is 3% of Rs. 768.00, i.e , Rs. 23.05. Therefore the net amount to be paid by him will be Rs. 768 00—Rs. 23 05, i.e , Rs. 744.95.

Now while teaching cash discount it will be useful if you collect some invoice forms from wholesale dealers Let the student study this form and understand the meaning of the different terms used in cash discount.

---

Q.1. What do you mean by cash discount ?

$$\frac{3}{10}, \quad \frac{n}{30}$$

Q.2. An invoice of Rs 2,100.00 carries the terms $4, \frac{3}{10}, \frac{1}{25}$ and $\frac{n}{45}$

Find the amount to be paid by the retailer :

(*i*) Immediately
(*ii*) On 5th day.
(*iii*) On 18th day.
(*iv*) On 30th day.

---

**Shares and Dividend**

Read the advertisement given by Benaras Hotels Ltd. (Hindustan Times dated 18th June, 1981) :

BENARAS HOTELS LIMITED

**(Incorporated under the Companies Act, 1956)**
**Registered Office . Fort, Ramnagar, Varanasi-221 003**

Public Issue of 6,00,000 Equity Shares of Rs. 10 each for cash at par

The above public issue was over-subscribed by more than 7 times. The Board of Directors of the Company thank the investing public for

their excellent support. The Company has, in consultation with the Delhi Stock Exchange, finalised the basis of allotment as follows :

| *No. of shares applied for* | *No. of shares allotted* | *No. of allottees* |
|---|---|---|
| 50 | 25 to 4 out of 9 applicants | 7316 |
| 100 | 25 to 2 out of 3 applicants | 5264 |
| 150-300 | 25 to each applicant | 7066 |
| 350-950 | 50 to each applicant | 1510 |
| 1000-1950 | 75 to each applicant | 382 |
| 2000 | 100 to each applicant | 23 |
| 4000 | 150 to each applicant | 11 |
| 8000-10000 | 250 to each applicant | 3 |
| | TOTAL | 21,575 |

There will be approximately 359 allottees for every 1 lakh of share capital.

Despatch of the refund orders and share certificates will commence shortly. All future correspondence in this regard should be addressed to Tata Consultancy Services, Lotus House, 6-New Marine Lines, Bombay 400 020.

*By order of the Board*

Benaras Hotels Ltd. is a company registered under Companies Act. In order to start a hotel, a large amount of money is needed A group of individuals get together and form a company. The top members form a Board of Directors. Look at the total investment made by Benaras Hotels Ltd It is Rs. 60,00,000. It is collected by public through 6,00,000 shares of Rs. 10 each. The total investment is called the Capital Stock. You can find that there are 6,00,000 shares of Rs. 10 each with Benaras Hotels Ltd. These shares have been purchased by the public. Each shareholder gets a share certificate which is called Stock Certificate. The pay value of a share is stated in this certificate. The company makes announcements in the newspapers for public issue of shares. It is open to all public. Any individual can purchase shares. Now read the following announcement made by the Andhra Valley Power Supply Company Ltd. (Hindustan Times, dated 8th June, 1981) ;

(This is only an announcement and not a prospectus)

THE ANDHRA VALLEY POWER SUPPLY COMPANY LIMITED

**(Incorporated on 31st August, 1916)**

**Regd. Office : Bombay House, 24 Homi Mody Street, Fort, Bombay : 400 023.**

Announcement regarding public issue of 5,25,00 Equity (Ordinary) Shares of Rs. 100/- each for cash at par.

Share Capital as on 30th September, 1980.

Authorised

| | | |
|---|---|---|
| 50,000 | 7% preference shares of Rs. 100 each | 50,00,000 |
| 60,000 | 11% cumulative Redeemable 'A' preference shares of Rs. 100 each | 60,00,000 |
| 10,25,000 | Ordinary shares of Rs. 100 each | 10,25,00,000 |
| 90,000 | Unclassified shares of Rs. 100 each | 90,00,000 |
| | TOTAL | 12,25,00,000 |

*Issued and subscribed*

| | | |
|---|---|---|
| 49,370 | 7% cumulative preference shares of Rs. 100 each fully paid | 49,37,000 |
| 44,435 | 11% cumulative Redeemable 'A' preference shares of Rs 100 each fully paid | 44,43,500 |
| 3,70,052 | Ordinary shares of Rs. 100 each fully paid | 3,70,05,200 |
| | Of the above Ordinary Shares 37,005 shares were issued as Bonus Shares by capitalisation of Eneral Reserve. | 4,63,85,700 |

*Present Issue*

| | | |
|---|---|---|
| 5,25,000 | Ordinary Shares of Rs. 100 each for cash at par. | 5,25,00,000 |

They have announced a public issue of 5,25,000 Equity (Ordinary) Shares of dimensions Rs. 100 each for cash at par. This is an annoucement not a prospectus. You can get a prospectus from the Company in which all information about the share and conditions for purchase have been given From the announcement we find that there are two kinds of shares—(*i*) preferred shares and (*ii*) common (ordinary) shares.

In the announcement of Andhra Valley Project you can see that there are 50,000 preferred shares of par value of Rs. 100—7% written before the preferred share means that *the preferred shareholder must receive 7% of the par value as annual dividend.* After deducting the dividend of the preferred shareholders, rest of the profit is divided in the form of dividend among common or ordinary shares. The method of calculation of the amount of dividend can be made clear by the following example .

**Example**

The Capital Stock of an oil company is Rs. 65 lakhs and is developed as below :

**Share Capital**

| *No. of shares* | *Type* | *Amount* |
|---|---|---|
| 2,500 | $5\frac{1}{2}$% preferred Stock shares of Rs 100 | 2,50,000 |
| 62,500 | Ordinary shares of Rs. 100 | 62,50,000 |
| | | 65,00,000 |

The net profit in a given year of this Company is Rs. 4,25,000 out of which Rs. 2,47,000 is to be distributed as dividend. Find out the amount .

(*a*) Mr. X will receive for his 50 shares of preferred stock.

(*b*) Mr. Y will receive for his 50 shares of common stock.

**Solution**

| | | |
|---|---|---|
| Total dividend declared by the Company | = | Rs. 2,47,000 |
| Preferred dividend on one share | = | Rs. $5\frac{1}{2}$% of Rs. 100 |
| | = | Rs. 5.50 |
| Total preferred dividend | = | Rs. 5.50 × 2500 |
| | = | Rs. 13,750 |
| Mr. X will receive on 50 shares | = | 5.50 × 50 |
| | = | Rs. 275 |
| Amount left for dividend on ordinary stock | = | Rs. 2,47,000 |
| (—) | | 13,750 |
| | = | Rs. 2,33,250 |

Dividend per share of ordinary stock

$$= \frac{2,33,250}{62,500}$$

$$= 3.732$$

Mr. Y will receive for 125 ordinary shares $= \text{Rs. } 3.732 \times 125$
$= 466.50$

---

Q. 1. Read newspapers and collect the announcement of public shares by different companies and corporations.

Q. 2. Prepare an exercise consisting of 10 questions based on the newspaper cuttings on share and dividend.

Q. 3. How do you calculate the profit on preferred stockholders? Is it calculated after finding out the profit of ordinary shares?

---

### Sales Tax and Local Taxes

Sales tax is realized on shoes, plastic articles, medicines, cosmetics etc. It is often stated as per cent. It is calculated on the list price. Look at the following cash memo :

**Cash Memo**

PRAKASH BOOT HOUSE

No. Dated : 10.10.80

Name : Gyan Singh

| | *Rs. np* |
|---|---|
| Shoe ambassador | 45 95 |
| S.T. 8% | 3.58 |
| TOTAL | 49.53 |

The sales tax on shoes is 8%. It is calculated on the list price (marked price).

Three types of problems as in commission and discount can also be identified in this case.

*Case 1* : Computation of sales tax from the given rate and list (marked) price.

*Case 2* : Computation of the rate from the given sales tax and list (marked) price.

*Case* 3. Computation of the list price from given rate and amount of sales tax.

By presenting examples on discounts, commission and sales tax simultaneously, the students should be able to understand that though the three cases of these topics are different, they are of the same nature and the process of computation is the same in all the topics. To make it more clear, examples can be presented as below

1. The sale price of a television set is Rs 4,000. A discount of 4% is allowed for cash payment Find the cash price of the television set.
2. A salesman's commission rate is 4%. He sells goods worth Rs. 4,000. How much commission does he receive?
3. Compute the sales tax on a television set that has a list price of Rs. 4,000, if the sales tax rate is 4%.

The examples should be set in such a manner so that the figures should be the same as given in the above examples

**Local Taxes**

In Municipal areas local taxes are also charged along with the sales tax Those taxes are calculated on the list price of the article and added to the price of the article for calculating its selling price. Let us take a problem.

**Problem**

On medicines 8% sales tax and 2% local taxes are charged. If the retail price of a medicine is Rs. 4.85p what will be its selling price?

**Solution**

$$\text{Sales tax} = 4.85 \times \frac{8}{100} = 0.39$$

$$\text{Local tax} = 4.85 \times \frac{2}{100} = 0.0970 \text{ or say } 0.10$$

$$\text{Selling price} = \text{Rs. } (4.85+0.39+0.10) = \text{Rs. } 5.34\text{p.}$$

*Note* · In calculating the amount, we can go only up to two places of decimal. So rounding off is required and the same has been done above.

**Compound Interest**

In previous classes, the students have learnt about simple interest. They have also learnt about compound interest and have solved prob-

lems on loan and growth of population. They have used the formula :

$$A = P\left(1 + \frac{r}{100}\right)^n \text{ for the computation of compound interest.}$$

Compound interest is also involved in instalment purchasing. It is also involved in purchasing a house from a Housing Board and fixed deposits.

Suppose, a house costs Rs. 42,000 and a down payment of Rs. 13,400 is made. The balance due is Rs. 28,600. Suppose the monthly payment is Rs. 440 and interest is 12%. Part of the payment is used to return the principal, and the rest of the payment is used for recovering the interest. At the end of the first month the interest to be paid by the buyer will be :

$$\begin{aligned} i &= p\,\frac{r}{100}\,t \\ &= 28{,}600 \times \frac{12}{100} \times \frac{1}{12} \\ &= \text{Rs. } 286.00 \end{aligned}$$

Of the first payment Rs. 286 is for interest and Rs. 440—286=154 is used to return the principal. The outstanding debit after one month is Rs. 28, 600—154 = 28, 446. When the second monthly payment is made, the buyer has to pay the interest on Rs. 28,446 for one month. So the interest is :

$$\begin{aligned} i &= \frac{28{,}446 \times 12 \times 1}{100 \times 12} \\ &= \text{Rs. } 284.46\text{p} \end{aligned}$$

Note that the interest is a little less for the second month. Out of Rs. 440, payment of Rs. 284. 46p is for interest and the remainder, i.e Rs. 440—284.46 = Rs. 155 54p is used to return the principal. The outstanding principal after the second payment is .

$$\text{Rs. } 28{,}446 - 155.54 = \text{Rs. } 28{,}290.46\text{p}$$

The process is repeated until the principal is paid in full. The student should observe that in early months of the contract most of the monthly payment is for interest and a relatively small amount is used to return the principal. Near the middle of the time about half of the deposit is for interest and the other half is to return the principal amount.

In newspapers and pamphlets from the Housing Boards there are advertisements for instalment buying. You may collect the informations and use them as teaching aids.

Look at the following problem. This is from a Housing Board's contract.

**Example**

A man purchases a house costing Rs 70,000 from the Housing Board. He pays Rs. 25,000 down payment. He deposits 120 monthly instalments of Rs. 775 to pay towards the outstanding principal and interest What is the rate of interest ?

**Solution**

This is an inverse problem of compound interest and requires the use of log tables. The same has been done as under

The balance due is Rs. 45,000

$$775\times120 = 45,000\left(1+\frac{i}{100}\right)^{120}$$

$$93,000 = 45,000\left(1+\frac{r}{100}\right)^{120}$$

$$120\log\left(1+\frac{r}{100}\right) = \log 93-\log 45$$

$$= 1.9685-1\ 6532$$

$$= .3153$$

$$\text{Log}\left(1+\frac{r}{100}\right) = \frac{.3153}{120}$$

$$= .00273$$

$$1+\frac{r}{100} = 1.007$$

$$\frac{r}{100} = 1.007-1 = .007$$

$$r = 0.7\%\text{ per month}$$

$$r = 0.7\times12 = 8.4\%\text{ per annum.}$$

*Note* Use of the logarithm table to solve the above problem simplifies calculation.

**Instalment Buying**

Radios, television-sets, refrigerators, and many other items are commonly sold on instalment contracts. The guiding factor in such transactions is the carrying charge rate. The carrying charge rate is not an interest rate. The equivalent interest rate is much greater than carrying charge rate This fact should cause no undue alarm. The carrying charge rate is calculated on the balance due for payment. It is added to

the payment due and the total is divided by the number of instalments to find out the amount of each instalment. Consider the following problem .

**Problem**

The cash price of a television-set is Rs. 4,500. The trade is valued at Rs 2,000. If the carrying charge rate is 12% find the monthly payment of a 12-month contract.

**Solution**

$$\begin{aligned}
\text{Balance due} &= 4{,}500-2{,}000 = 2{,}500.00 \\
\text{Carrying charges} &= 12\% \times 2{,}500 = 300.00 \\
\text{Monthly payment} &= \frac{2{,}500+300}{12} \\
&= \frac{2{,}800}{12} \\
&= 233.33 \\
&= \text{Rs. } 233.35 \text{ (after rounding off)}
\end{aligned}$$

*Note* : 1. Rounding off is a convention in trade.
2 Efforts should be made to bring in "live" data.

---

Q. Socrates can teach any concept through questioning only.

Prepare a sequence of questions to elicit the answer of the following problem :

*Problem* : A certain sum of money is invested at 5% compounded half yearly. In how many years will the amount double itself ?

---

**Taxes**

The Central and State Governments prepare Five Year Plans for the welfare of the nation. Under these plans, the Government decides to take up various developmental and welfare programmes. An example of such a welfare programme is universalisation of primary education of 6-14 years old children. To implement this the Government has to bear expenditure on education. The question now arises as to which are the sources to provide this money. "Taxes" is one of the sources.

Government realizes income tax from those persons whose income is above certain amount, say, Rs. 10,000 per annum. Amount of income

| **Total Income Slab** | | **Tax payable for every individual HUF (except those as per col. 3) URF, AOP, Bodies of Individuals (whether incorporated or not) and Artificial Judicial Persons preferred to in section 2 (31) (vii) of the Income Tax Act)** | | | **Tax payable for every HUF having at least one member whose total income exceeds Rs 10,000** | | |
|---|---|---|---|---|---|---|---|
| 1 | | 2 | | | 3 | | |
| *From* | *To* | *Fixed amount for the sale* | *Plus* | *Variable amount as under* | *Fixed amount for the slab* | *Plus* | *Variable amount as under* |
| Up to | 10,000 | Nil | | | | | |
| 10,001 | 15,000 | Nil | Plus 15% | exceeding Rs. 8,000 | Nil | Plus 18% | exceeding Rs. 8,000 |
| 15,001 | 20,000 | 1,050 | Plus 18% | exceeding Rs. 15,000 | 1,260 | Plus 25% | exceeding Rs. 15,000 |
| 20,001 | 25,000 | 1,950 | Plus 25% | exceeding Rs 20,000 | 2,510 | Plus 30% | exceeding Rs. 20,000 |
| 25,001 | 30,000 | 3,200 | Plus 30% | exceeding Rs. 25,000 | 4,010 | Plus 40% | exceeding Rs. 25,000 |
| 30,001 | 50,000 | 4,700 | Plus 40% | exceeding Rs. 30,000 | 6,010 | Plus 50% | exceeding Rs. 30,000 |
| 50,001 | 70,000 | 12,700 | Plus 50% | exceeding Rs 50,000 | 16,010 | Plus 55% | exceeding Rs 50,000 |
| 70,001 | 1,00,001 | 22,700 | Plus 55% | exceeding Rs 70,000 | 27,010 | Plus 60% | exceeding Rs. 70,000 |
| 1,00,001 | and more | 39,200 | Plus 60% | exceeding Rs. 1,00,000 | | | |

Provided that in case where total income does not exceed Rs. 10,540 the Income Tax payable shall not exceed 70% of the total income exceeding Rs. 10,000.

The amount of Income Tax computed in accordance with the above Tax Table shall be increased by a surcharge at 15% thereof

tax paid by a person depends on his income.

The computation of income tax involves a knowledge of the income tax laws but the fundamental principles can be discussed in secondary classes. Simple income tax forms can be discussed. The chart on page 41 may be used for computation of income tax.

*Note* : The rate of income tax changes quite often. Hence an up-to-date chart should be used while calculating the tax.

**Problem**

A man's income for 1979-80 was Rs. 20,000. His deductions amounted to Rs. 3,000 for CPF and CTD He gets the usual exemption for Rs. 3,500. Find the taxable income and calculate the income tax from the above chart assuming the man to be a member of a joint family.

**Solution**

His taxable income = 20,000 —3,000—3,500 for 1979-80
= 13,500.00

He is in the slab of 10,001 to 15,000. Since he is a member of joint family, income tax will be calculated according to column 3.

Amout exceeding Rs. 8,000 = 5,500

Income-tax $= \dfrac{5{,}500 \times 15}{100}$

$= 825.00$

Surcharge = 10% of Rs. 825.00

= 82 50 = 83.00

Total income-tax payable = 825.00+83 00

= 908.00

Q Calculate the income tax deduction of your friends using the following form :

Income tax deduction from the salary of individuals during 1980-81

1. Total salary during 1980-81
2. Honorarium and other remuneration received during 1980-81
3. Arrears of pay/ADA/CDS received during 1980-81.
4. Gross income.

**Deductions**

1. House rent for which rebate is sought
   Total income .

2. Standard deductions :
   20 per cent of salary up to 10,000 and 10% of salary in excess thereof subject to a maximum of Rs. 3,500
   Gross total income ·
3 Contributions towards L.I C./GPF/CPF/ULIP etc.
   whole of 5,000, 50% of next 5,000 and 40% of the balance subject to 30% of the estimated salary after allowing standard deductions of Rs. 3,000 whichever is less
   Net taxable income to be rounded off to the nearest rupee.

1. Income tax up to Rs. 12,000
2. Rs. 8,000 to 15,000
   15% of the amount by which income exceeds Rs 8,000
3. 15,001 to 20,000 @ Rs 1050 + 18%
   of the excess of Rs 15,000
4. 20,001 to 25,000 @ Rs. 1,950 + 25%
   of the excess of Rs 20,000.
5. 25,001 to 30,000 @ Rs. 3,200 + 30%
   of the excess of Rs. 25, 000.

*Note* : Income tax is always paid in whole rupees. Hence the surcharge has been rounded off to the next rupee.

**Savings Bank Account**

The Savings Bank Account can be opened in any post office by depositing Rs 5/- or above Post office will give a pass book in which the amount deposited will be entered Money can be deposited on any working day of the week, but the amount can be withdrawn only once in a week. Interest will be paid on the deposit The interest is calculated on the minimum amount which remains deposited between 6th day and the last day of the month The rate of interest at present is $5\frac{1}{2}\%$ per annum. The interest is calculated and entered on 31st March

1. Post office interest is free from income tax.
2. Any person can deposit up to Rs. 35,000 in post office savings bank account.
3. The rate of interest changes from time to time.

**Problem**

Let us look at a page of a Pass Book of an account holder having his account in a post office savings bank and calculate the interest. Following are the entries in a page of a post office savings bank pass book.

| *Date* | *Deposit* | *Withdrawal* | *Balance* | *Sig. of the post master* | *Remarks* | *Stamp with date* |
|---|---|---|---|---|---|---|
| 1.4 79 | 4,000.00 | — | 4,000.00 | — | B E. | |
| 5.6.79 | 500.00 | — | 4,500 00 | — | Cash | |
| 8.7.79 | — | 1,000.00 | 3,500 00 | — | — | |
| 9.9.79 | 700.00 | — | 4,200.00 | — | Cash | |
| 9 1.80 | — | 200.00 | 4,000 00 | — | — | |
| 1.4.80 | ? | — | ? | — | Interest. | |

Calculate the interest to be entered on 1.4.80 in the pass book Rate of interest is $5\frac{1}{2}\%$ per annum.

*Note* : Interest in the post office savings bank pass book is entered on 1st April, 1980 while the interest is calculated up to 31st March .

**Solution**

| | | |
|---|---|---|
| Principal for the month of | April | 4,000.00 |
| | May | 4,000.00 |
| | June | 4,500.00 |
| | July | 3,500.00 |
| | August | 3,500.00 |
| *Note* : The minimum balance of each month is added to find out the principal of one month and then the interest is calculated as per formula. | September | 3,500.00 |
| | October | 4,200.00 |
| | November | 4,200 00 |
| | December | 4,200.00 |
| | January | 4,000.00 |
| | February | 4,000.00 |
| | March | 4,000.00 |
| Total principal for one month | | 47,600.00 |

$$\text{Interest} = \frac{P \times R \times T}{100}$$

$$= \frac{47{,}600 \times 11/2 \times 1/12}{100}$$

$$= \frac{47{,}600 \times 11 \times 1}{2 \times 12 \times 100}$$

$$= \frac{1,309}{6}$$

$= 218.15$ (corrected up to 5 paise)

Q. Calculate the interest for the year 1979-80 from the entries of Rampal's Pass Book as below .

| *Month* | *Deposit* | *Withdrawals* | *Balance* | *Particulars* | *Stamp with date* |
|---|---|---|---|---|---|
| 1.4.79 | 6,600.00 | — | 6,600.00 | B.F | |
| 3.4.79 | 1,500.00 | — | 8,100.00 | Cash | |
| 18.6 79 | — | 2,400.00 | 5,700.00 | — | |
| 27.8.79 | 3,600.00 | — | 9,300.00 | Cash | |
| 6.1 80 | — | 500.00 | 8,800 00 | — | |
| 2 3.80 | — | 1,800.00 | 7,000.00 | — | |
| 1.4.80 | ? | — | ? | Interest | |

Savings Bank Account can also be opened in a bank similar to the post office. The interest in the bank is calculated twice in a year, i.e., on 30th June and 31st December and added to principal while in the Post Office it is only calculated once, i.e., 31st day of March.

*Note* : The bank provides the facility of cheque book. Anyone who wants to avail of this facility must have a minimum of Rs. 100/- in the pass book. On using the cheque facility the rate of interest is reduced by 1%

**Problem**

Look at a page of Savings Bank Account of a Pass Book and calculate the interest.

The rate of interest is 4% per annum.

| *Date* | *Particulrs* | *Withdrawn* | *Deposited* | *Balance* | *Signatures* |
|---|---|---|---|---|---|
| *1980* | | | | | |
| Jan. 5. | Cash | — | 10.00 | 10.00 | |
| Jan. 7. | Cash | — | 1,000.00 | 1010.00 | |
| Feb. 2 | Self | 400.00 | — | 610.00 | |
| Feb. 5. | — | — | 500.00 | 1110.00 | |
| March 8. | Cash | — | 210.00 | 1320 00 | |
| May 3 | Cash | — | 100.00 | 1420.00 | |
| June 4. | Self | 400.00 | — | 1020.00 | |
| July 1. | Interest | — | ? | ? | |

**Solution**

| | | |
|---|---|---|
| Principal for the month of January | | — |
| | February | 1110.00 |
| | March | 1110.00 |
| | April | 1110.00 |
| | May | 1420.00 |
| | June | 1020.00 |
| Total principal for one month | | 5770.00 |

$$\text{Interest up to June} = \frac{P \times R \times T}{100}$$

$$= \frac{5770 \times 1/12 \times 4}{100}$$

$$= \text{Rs. } 19.23$$

$$= \text{Rs. } 19.20\text{p}$$

**Teaching Strategies**

We should use actual business forms to make the teaching of application of percentage more interesting and more meaningful An actual pass book of a savings bank account deposit, a fixed deposit bond, an income tax form, a chart for calculating income tax, a sales tax chart used in variety stores, an invoice form, etc., will make the study of percentage more meaningful The advertisement from a local newspaper can be used for problems on instalment buying, shares & dividends and discounts. You can also collect tables, which are used to calculate total amount payable at maturity of recurring deposits

Visit to the post office and lectures by income tax officers, commission agents should be arranged so that the students can get the first hand knowledge.

Most of the problems on profit and loss, interest, etc , involving percentage are of direct nature up to Class VIII. Students substitute the given data in the formula to get the answer. Here you should maintain a diagnostic point of view There are some points where students are more prone to commit mistakes. For example :

(A) In calculating % profit or % loss, some students calculate profit or loss on selling price or marked price instead of cost price.

(B) In double transactions, the students use C.P. of first transaction to calculate profit or loss in both the cases.

Such errors are committed by those students who often try to search out clues to classify problems. We must discourage this and allow the students to follow the steps of problem solving. (See page 19).

Let us illustrate it as follows :

**Problem**

The cost of a book is 75 paise plus 75% of its total cost. What is the price of the book?

**First** . Understanding the problem :

| | |
|---|---|
| What is unknown? | Price of the book. |
| Comprehension | Total cost and price of the book are the same, which is the cost price of the purchaser. |

**Second** : Devising a plan ·

(1) Total cost of the book is its price and this is the same as marked price.

(2) Translate the problem and form an equation.

**Third** . Carrying out the plan.

$75p + 75\%$ of total cost $=$ Price of the book

$75p + 75\%$ of $x = x$

Where Rs. $x$ is the total cost of the book.

or $$75p + \frac{75}{100}x = x$$

or $$300p + 3x = 4x$$

or $$x = 300p = \text{Rs. } 3.00$$

**Fourth** : Examine the solution obtained ·

(1) We can say that 25% of the marked price is equal to $75p$. Therefore, price of the book is $75p \times 4$, i e , Rs. 3.00

(2) We can construct similar problems. For example, if 50% of the cost plus $50p$ is equal to total cost, what is the price of the book?

Give some problems to the class and ask the students to solve them. Now you enquire how they thought for the solution. Lay more emphasis on the mental process of their thinking

**Evaluation**

The objectives of teaching Commercial Mathematics are purely utilitarian. It enables an individual ·

(1) to check the cash memos and detect possible errors (if any) while shopping.

(2) to compare the market prices and find out which is a better bargain for him.

(3) to calculate the interest on his savings and investments.

(4) to compute different types of taxes.

(5) to study market fluctuations.

(6) to understand the hire and purchase bargains and the instalment plan scheme

**Specimen Test Items**

1. If '$a$' is greater than '$b$' by 10% and '$b$' is greater than '$c$' by 10%, then by what per cent is '$a$' greater than '$c$' ?
2. If water contains $11\frac{1}{9}\%$ of hydrogen by weight, determine the weight of hydrogen in 639 gms. of water.
3. If a man pays an instalment of Rs 10 in a recurring deposit for 61 months, he gets Rs. 797.40p. Calculate the rate of compound interest.
4. A refrigerator is offered for sale at Rs. 8,000.00 less successive discounts of 20% and 5%. What is the sale price of the refrigerator?
5. A fruit seller buys oranges at 3 for Re 1.00 He will sell them at 5 for Rs. 2. In order to make a profit of Rs. 20 how many oranges will he sell?
6. 'A' owns a house worth Rs 50,000.00. He sells it to 'B' at 10% profit. 'B' sells the house back to 'A' at a loss of 10%. Calculate the profit of 'A'.
7. If the population of a city is 10,000 find its population at the end of 3 years if the population increases every year by 10% of what it is at the beginning of the year.
8. A house is sold for Rs. 48,000.00 cash or Rs. 28,000.00 as down payment and Rs. 700 a month for 42 months. Determine the rate of interest.
9. Calculate the interest for the year 1979-80 from the entries of Savings Bank pass book of a post office as below :
   (rate of interest is 5%)

| *Date* | *Deposit* | *Withdrawal* | *Balance* | *Signatures* | *Remarks* | *Stamp with date* |
|---|---|---|---|---|---|---|
| 1.4.79 | 5,800 | — | 5,800 | | B.E. | |
| 3.4.79 | — | 2,500 | 3,300 | | — | |
| 9.8.79 | — | 1,300 | 2,000 | | — | |
| 14.10.79 | 7,000 | — | 9,000 | | Cash | |
| 21.3.80 | 2,000 | — | 11,000 | | Cash | |
| 1.4 80 | ? | — | ? | | Interest | |

10. Below is the share capital of A.B.C. Limited.

| | | |
|---|---|---|
| 4,00,000 | $7\frac{1}{2}\%$ Cumulative Preference shares of Rs. 100 each | Rs. 4,00,00,000 |
| 1,00,000 | 11% Cumulative Preference shares of Rs. 100 each. | Rs. 1,00,00,000 |
| 16,25,000 | Ordinary shares of Rs. 100 each. | Rs. 16,25,00,000 |

The Company declares a dividend of Rs. 1,55,00,000. If you have 500, $7\frac{1}{2}\%$ Cumulative preferred shares, 200, 11% Cumulative preferred shares, and 5,000 Ordinary shares, find the amount of dividend you receive.

11. What cash payment will settle a bill for 250 bags of cement at

Rs. 20. 20 per bag less 10% and terms cash 5, $\frac{n}{30}$?

**Assignments for Student Teachers**

1. A farmer sold his buffalo for Rs. 2200 at a gain of 10%. Find the cost price of the buffalo.

Following is the solution of the above problem as given by a student :

| | |
|---|---|
| Gain on Rs. 100.00 | = Rs. 10.00 |
| Gain on Rs. 2,200 00 | = Rs. 220.00 |
| ∴ Cost price | = 2,200—220 |
| | = Rs. 1,980.00 |

Point out the mistake (if any) in the above solution and suggest the steps you would take to overcome the mistake.

2. An electric dealer offers a washing machine for Rs. 750 less successive discounts of 10% and 4%. Find the selling price of the washing machine.

Following is the solution given by a student :

| | |
|---|---|
| List price | = 750 00 |
| 1st Discount | $= \frac{10 \times 750}{100}$ = Rs. 75.00 |
| Balance after Ist Discount | = 750−75 = Rs. 675.00 |
| 2nd Discount | $= \frac{4 \times 750}{100}$ = Rs. 30.00 |
| Balance after 2nd Discount | = 675—30 = 645.00 |
| Selling price | = Rs 645 00 |

Check whether the above solution is correct or not? If not, suggest

the steps to be taken by you so that the mistake is corrected.

3. A fruit-seller sold apples at Rs 5 per kg thereby making a gain of 25% Find the cost per kg of the apples.

(a) Reconstruct the above problem so as to find the selling price.

(b) Reconstruct the same problem so as to find the gain per cent.

(c) Write down the relationship between C.P., S.P. and gain % and check whether it holds good in all the above three cases.

4. The marked price of a textbook is Rs 5. The book-seller offers 5% cash discount. Find the price at which the book is sold?

(a) Construct the above problem so that student has to find the cash discount

(b) Reconstruct the same problem so that the student has to find the marked price.

(c) Write down the relationship between the marked (list) price, discount and selling price. Check whether this relationship holds in the above three cases.

5. The population of a country is said to increase at the rate of 2% per annum.

With what topic in arithmetic would you associate this mode of growth of population ?

If the present population be 4,00,000 what shall it be at the end of 4 years? (Use log-table to simplify the calculation).

(6) The list price of a watch is Rs. 625.00 The watch dealer offers it for sale under the instalment plan for Rs 85.00 cash down and the rest in six equal monthly instalments of Rs 100. Would you suggest purchasing the watch under the instalment plan scheme ? Bring out the strong points of your suggestions.

Also calculate the rate of interest in the above problem.

7. You are asked to develop a lesson plan on hire-purchase mode of payment under the instalment plan. Based on the problem No. 6, frame suitable questions to be asked from the students so as to ensure that they have grasped the problem.

8. How will you introduce a topic on insurance. Bring out the basic idea involved in all types of insurance.

CHAPTER 3

# Statistics

## Introduction

When you want to compare the achievement of the pupils of your school with that of the pupils of other schools in a given public examination, we collect the marks obtained by the pupils in the said public examination In statistics we call these marks as *data.* When the marks (data) are available, we can go for satatistical analysis to reach some interpretation. During statistical analysis the data pass through several stages·

(1) Classification of data

(2) Presentation and comparison of data through charts and graphs.

(3) Analysis of data through computation.

(4) Interpretation of data.

We will discuss in this lesson how to develop and teach these stages in a class.

## Content Covered in the Unit

(*i*) Classification of data :

(*a*) Class intervals.

(*b*) Tabulation of data.

(*ii*) Diagrammatic representation :

(*a*) Introduction

(*b*) Bar Diagram.

(*c*) Area Diagram.

(*d*) Pictogram.

(*e*) Pie Diagram.

(*f*) Histogram.

(*g*) Frequency Polygon.

## Classification of Data

(*a*) Class intervals

Let us measure the heights of the students in one class (say, of strength 50) ;

Heights of 50 students in centimetres are given below :

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 136 | 146 | 141 | 159 | 137 | 142 | 143 | 127 | 151 | 131 |
| 142 | 136 | 143 | 133 | 146 | 132 | 147 | 148 | 150 | 132 |
| 143 | 121 | 135 | 126 | 135 | 144 | 128 | 152 | 141 | 131 |
| 127 | 139 | 144 | 151 | 141 | 154 | 138 | 142 | 148 | 135 |
| 140 | 142 | 156 | 136 | 143 | 155 | 140 | 149 | 133 | 134 |

First we observe that the height is measured correct to a centimetre Next we note that 121 cms. is the smallest item and 159 cms. is the largest item All the other items lie in between 121 and 159, i e , in the span of 38 cms.

Let us now group the heights into various classes, say, the students whose heights are within 120-125, 125-130, 130-135, 135-140, 140-145, 145-150, 150-155, 155-160. As the smallest item is 121 cms. we choose the smallest class-interval to be 120-125. Similarly the largest item being 159 the last class-interval is 155-160.

Now we come across a trouble : where should we include the student with height 145 cms. in 140-145 or 145-150 ? To avoid this difficulty we can write the class-intervals as 120 5-125.5, 125.5-130.5 etc. so that 145 will be included in 140 5-145.5 and not in 145.5-150.5.

Also to avoid the ambiguity for deciding where the height 145 cms. will be in 140-145 or 145-150 ? We have the proposition that the UPPER LIMIT IS COUNTED IN THE SUCCEEDING interval. According to this convention we have to include 145 in 145-150 and not in 140-145. Thus we have two types of class intervals.

| *A* | *B* |
|---|---|
| 120-125 | 120.5-125.5 |
| 125-130 | 125.5-130.5 |
| 130-135 | 130.5-135.5 |
| 135-140 | 135.5-140.5 |
| 140-145 | 140.5-145.5 |
| 145-150 | 145.5-150.5 |
| 150-155 | 150.5-155.5 |
| 155-160 | 155.5-160.5 |

The mid-point of each class-interval is the *class mark*. 143 is the class mark of 140.5-145.5. The class mark of 140-145 is 142.5. Here the length of each class-interval is 5 and the number of class-intervals is 8. The class-intervals are also sometimes written in the reverse order, taking the last class-interval as first and so on.

We have seen that the heights of the students have been grouped into 8

classes. The class interval in each case is 5, if the length of the class-interval is decreased, the number of classes will increase. Now a question arises how to decide suitable class limits. We must keep the following points in mind while deciding class limits and class intervals :

1. Class limits must be fixed with reference to accuracy of data.
   If the height is measured correct to 1 mm, it will be useful to take class limits as 120.5-125.5, 125.5-130.5 etc. If the height is measured correct to 1 cm, it will be better to take class limits as 120-125, 125-130, etc.
2. The number of classes should neither be so large as to make the grouped data look very small, nor so small as to make it unwieldy. Note the smallest and largest items and find their difference. The difference may be divided by a convenient length to obtain the number of class intervals. Make it a whole number and decide the class limits.
3. The length of each of the class intervals should be the same.
4. For a fairly large data (say more than 100) the possible number of class intervals can be between 10 and 25. If the data are 50 or so near about, the number of class intervals may be between 5 and 10
5. Every class interval should be defined precisely. We should not include class limits as :

| Age |
|---|
| *Under 10* |
| 10-20 |
| 20-30 |
| 30-40 |
| 40-50 |
| Above 50 |

Under and above are not precisely defined.

---

Q.1. Find class marks of 125-130 and 125.5-130 5 on a number line. Is the class mark same for both intervals ?

Q.2. Decide suitable class intervals for the following data :

| | | | | |
|---|---|---|---|---|
| .103, | .15, | .170, | .132 | .151, |
| 17, | .121 | .113 | .120 | .152 |
| .114 | .133 | .125 | .164 | .156 |
| .165 | .108 | .185 | .132 | .138 |

Q.3. Construct a multiple choice type question :

(i) to test the understanding of the proposition for deciding in which class interval 145 will lie, 140-145 or 145-150 ?

(ii) to test the comprehension of precaution 2 on page 53.

---

**Tabulation of Data**

In classifying the data we write the class intervals one below the other. The given data (height of students) can be put into classes :

(i) by arranging them in ascending or descending order.

OR

(ii) by using tally mark

(i) *Arranging data in ascending order*

If the number of data is less (nearly 20), we can arrange them in ascending order easily If the number of data is large, it takes much time in arranging them in ascending order and also there are greater chances of committing errors

Arrange the height of the students on page 52 in ascending order.

(ii) *Use of Tally Marks* :

In classifying we write the class-interval one below the other. We take one number after another. Mark a stroke (like / ) against the class to which it belongs. The 'stroke' is called a 'tally'. To facilitate counting we mark the fifth stroke after every four (////) diagonally over the four like ~~////~~ . Finally the total number of items in that class is counted and written against the class concerned. It is called the frequency of the class The tabular form consisting of the class intervals and their frequencies is called **frequency distribution**.

For the data (heights of students) on page 52, let us form the frequency distribution.

**Heights of Students in Centimetres**

| *Class interval* (1) | *Tally mark* (2) | *Frequency* (3) |
|---|---|---|
| 1. 120.5-125.5 | / | 1 |
| 2. 125.5-130.5 | //// | 4 |
| 3. 130.5-135 5 | ~~////~~ ~~////~~ | 10 |
| 4. 135.5-140.5 | ~~////~~ /// | 8 |

| | | |
|---|---|---|
| 5. 140.5-145.5 | 卌 卌 //// | 14 |
| 6. 145.5-150 5 | 卌 / | 6 |
| 7. 150.5-155.5 | 卌 | 5 |
| 8. 155.5-160.5 | // | 2 |
| Total | | 50 |

We must follow the general rules for the tabulation of data to prepare the frequency distribution as given below :

1. The title should be written on the top of the table.
2. Columns and rows should be numbered to facilitate referring to the table.
3. Heading of the columns and rows should be without ambiguity.
4. The tables should be accurate, attractive, neat and tidy.

---

Q.1. Prepare a frequency distribution of the following data giving the index number of 30 commodities in a certain year.

94, 95, 96, 96, 96, 97, 99, 99, 100, 100, 101, 101, 102, 104, 104, 104, 105, 106, 106, 107, 108, 108, 109, 109, 110, 110, 111, 112, 113, 128.

Q 2. Do you mark tally for the above data to form frequency distribution table ? Give reasons in support of your answer.

Q.3. When do we mark tally to prepare a frequency distribution table ?

Q.4. "Students must try to arrange heights of fifty students in ascending order". What is the purpose of this activity ?

---

## Diagrammatic Representation

### Introduction

The purpose of tabulation is to arrange masses of unwieldy data according to certain characteristics. The objective of diagrams is to illustrate these characteristics. They are useful aids for explanation, exposition and interpretation of data. Diagrams present dry and uninteresting statistical facts in the shape of attractive and appealing pictures and charts. They give a bird's eye view of the whole mass of statistical data. The impression created by a diagram or a picture is likely to last longer in the mind than the effect created by a mass of numbers. But we should remember that the diagram shows only a limited amount of approximate information.

There are many methods of pictorial presentation of numerical data. We will discuss some of these below.

**Bar Diagram**

The term 'Bar' is used for a thick wide line. The width of the bar is arbitrary and is used merely to make the diagram look more attractive, beautiful and explanatory. Let us look at the bar-diagram of the following data obtained from plan outlay of the Central Government under the fourth Five Year Plan.

| *S. No.* | *Head* | *Rupees (in crores)* |
|---|---|---|
| 1. | Transportation & communication | 3237.36 |
| 2. | Irrigation and flood control | 1086.57 |
| 3. | Agriculture and allied sectors | 2723.18 |
| 4. | Industry & minerals | 3337.71 |
| 5. | Education | 822.66 |

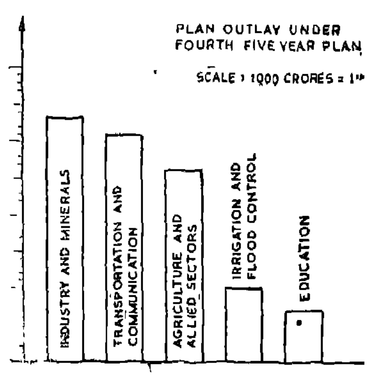



Fig. 3 1

Here there are no class intervals. Expenditure is mentioned in different 'HEADS'. Bars of uniform width with equal intervening spaces have been drawn on a common base after the scale has been adjusted in proportion to the magnitude of the largest item. In the above bar diagram, one inch represents 1,000 crores of rupees Is it possible to find out from bar-diagram that expenditure on Transport and Communication is 3237.36 crores of rupees ? We can only get approximate but not actual expenditure from the bar-diagram, but we can compare the expenditure on different items. We can say maximum expenditure is on industry and minerals.

What can we do to get a better approximation? We can choose scale as large as possible depending on the size of the paper.

The bar diagram may also be used to exhibit the division of a whole (head) into its component parts. If a given magnitude (head) can be split up into further sub-divisions or if there are different quantities forming the sub-divisions of the different total, simple bars may be sub-divided in the ratio of the various sub-divisions of the heads to exhibit the ralationship of the parts to the whole.

Look at the sub-divided bar diagram for the following data.

**Increase in Labour Force During 1961-62**

| *State* | *Labour Force in 1961* | *Labour Force in 1962* |
|---|---|---|
| Andhra Pradesh | 16.13 | 20.89 |
| Kerala | 5 51 | 8.46 |
| Karnataka | 9.40 | 13 15 |
| Tamilnadu | 13.99 | 18.09 |

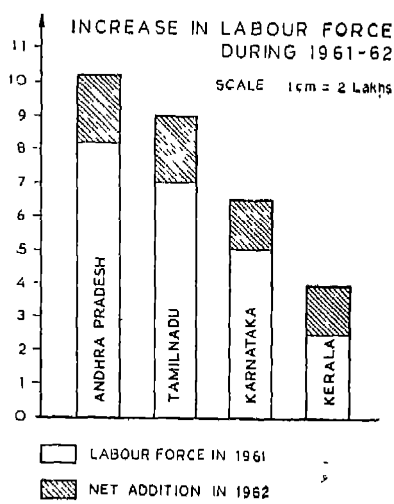



Fig. 3.2

If the sub-divisions are large in number, the sub-divisions may be reduced to percentages of the whole Then the height of the bar will represent 100 and the other components in percentages may be represented by the sub-divisions of the bar. We call this percentage Bar Diagram.

Look at the percentage bar diagram for the following data :

**Sex Distribution of Population in three Cities**

| *Cities* | *Males* | | *Females* | |
|---|---|---|---|---|
| | *Actual* | *Percentage* | *Actual* | *Percentage* |
| Lucknow | 2,78,605 | 56.1% | 2,18,257 | 43.9% |
| Agra | 2,06,459 | 54.8% | 1,67,206 | 45.2% |
| Allahabad | 1,85,113 | 55.9% | 1,47,182 | 44.1% |

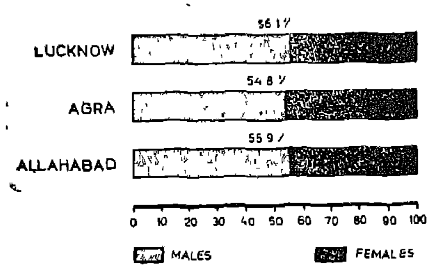



Fig. 3.3

If there are two or more independent quantities, say, the number of students in Classes IX, X and XI respectively, we use double or triple bars for comparison.

Look at a triple bar diagram for the following data :

**Number of Students in Different Classes of the Demonstration School for the Years 1964, 1965 and 1966**

| *Year* | *IX* | *X* | *XI* |
|---|---|---|---|
| 1964 | 350 | 300 | 250 |
| 1965 | 300 | 250 | 250 |
| 1966 | 275 | 225 | 200 |

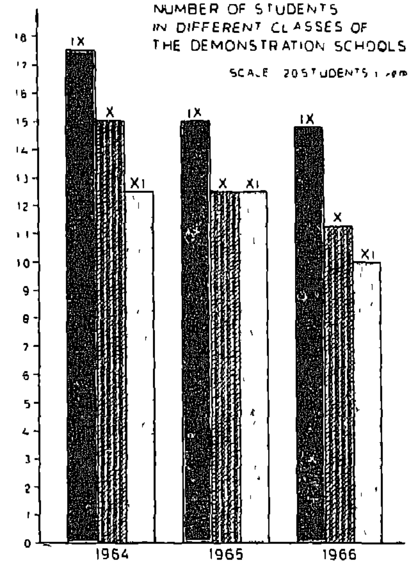



Fig 3.4

While drawing the diagram, we must select a scale according to the size of the paper. You can see that double bars or triple bars are very useful for direct comparison between two or more independent quantities. It will be useful if you ask the student to prepare bar-graph depicting various school informations. You can also ask your students to find the places where such graphs can be seen.

---

Q.1. "Practice is necerssary for selecting scale". Justify.

Q 2. While drawing the bar-graph of number of students in different classes in the above example in different years, is it necessary to start from 0 ? Can we not start from 200 ? Suggest some suitable starting point.

---

## Rectangular Diagram

You have seen that in bar-diagram the width of the bar has no significance with the data. If we take width into account we will have rectangular diagram The area of each rectangle will represent the corresponding magnitude. Rectangular diagrams are used when two or more quantities are to be compared and each is sub-divided into several components Suppose we have to draw a rectangular diagram to show the expenditure on the same items in two family budgets with

different incomes, then rectangles can be drawn with income as the width of the rectangle and 100 as height of each rectangle. The several items of expenditure may be reduced as percentages and represented on the rectangles.

Look at the rectangular diagram for the data given below.

The following table gives details of the monthly expenditures of three families.

| *Items of expenditure* | *Family A in Rs.* | *Family B in Rs.* | *Family C in Rs.* |
|---|---|---|---|
| Food | 240.00 | 500 00 | 600 00 |
| Clothing | 50.00 | 160.00 | 200.00 |
| House rent | 40.00 | 100.00 | 300 00 |
| Education | 20.00 | 100.00 | 150 00 |
| Miscellaneous | 100.00 | 200 00 | 300.00 |
| TOTAL | 450.00 | 1,060.00 | 1,550.00 |

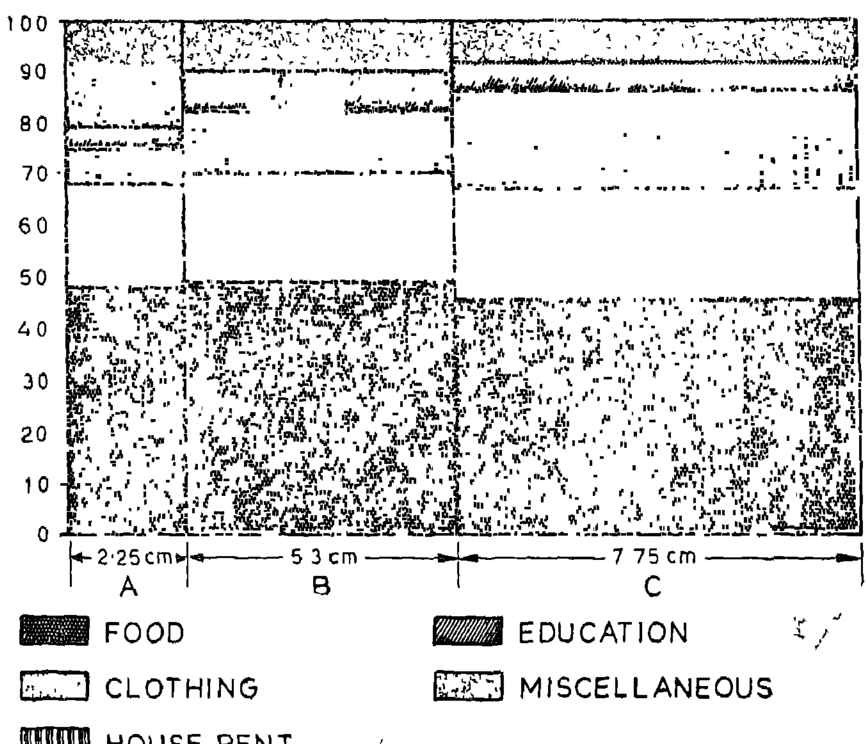



Fig. 3.5

Q 1. What is the difference between the bar-diagrammatic and rectangular diagrammatic representation?

Q.2 When do we use rectangular diagrammatic representation?

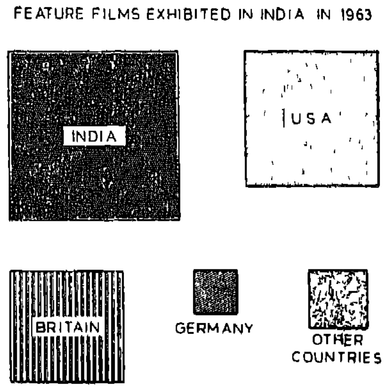



Fig. 3.6

**Square Diagram**

Among two-dimensional diagrams, sometimes squares give a better comparison than rectangulars and bars. They are specially useful when some items of the series have values much higher than the others.

In the construction of squares first of all the square root of the various numbers representing the data is calculated and then squares are drawn with the lengths of their sides in the same proportion as the square root of the original figures since the area of a square is side $\times$ side.

Look at the square diagram for the following data. The table gives the country of origin of feature films exhibited in India in 1963 :

| *Country* | *India* | *USA* | *Britain* | *Germany* | *Other countries* |
|---|---|---|---|---|---|
| No of films | 144 | 81 | 64 | 9 | 16 |

| *Country* | *No. of films* | *Square root* |
|---|---|---|
| India | 144 | 12 |
| U.S A. | 81 | 9 |
| Britain | 64 | 8 |
| Germany | 9 | 3 |
| Other countries | 16 | 4 |

(See the above diagram)

**Pictogram**

We can represent the numerical data by means of appropriate pictures. The number of pictures drawn or the size of the different pictures are in proportion to the values of the different items. So in drawing pictures it should be borne in mind that the proportion in which the natural objects are found should not be disturbed. For instance, if the number of cows in India, U.S.A. and China are in the ratio 5:2:4, we should draw pictures of 5 cows representing their number in India, 2 cows for U.S.A. and 4 for China.

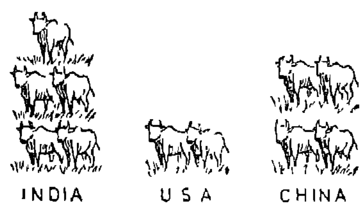



Fig 3.7

The ratio 5:2·4 is the same as 5/2.1:2. Here we cannot represent two-and-a-half cows for India, because cows cannot be found half in nature. This point should be stressed while teaching pictogram.

---

Q. If the ratio of population of men and women of a country is 2/3·1, what simple ratio you will consider to show it by pictogram?

---

**Pie Diagram**

To understand pie diagram the student must know :

(*i*) The angle at the centre of a circle is 360°.

(*ii*) Area of the sectors of a circle are proportional to the angles subtended by them at the centre.

(*iii*) Division into proportional parts.

We have divided a bar and a rectangle to show the components of the data. In the case of pie diagram this can be easily done, as the angles subtended by them at the centre are proportional to the component parts. Taking this proportion we can draw the sectors to represent the different component parts. We can make it clear by presenting a few pie diagrams before the students.

To teach the students to draw a pie diagram, we should do some exercises on the blackboard in the class with the help of the students,

For example .

Suppose we are given the time-table of the class. Let the circle represent 48 periods of a week. The following table gives the time allotted for different subjects during a week :

**Time-table of Class X**

| *Subject* | *No. of periods* | *Degrees of sectors* |
|---|---|---|
| 1. Hindi | 9 | 67.5° |
| 2. English | 6 | 45.0° |
| 3. Gujarati | 3 | 22.5° |
| 4. Gen Science | 9 | 67.5° |
| 5 Mathematics | 5 | 37.5° |
| 6 Social Studies | 4 | 30 0° |
| 7. History | 9 | 67 5° |
| 8. Phy. Education | 3 | 22.5° |
| Total | 48 | 360.0° |

Here the circle represents the total periods in a week, i.e , 48 periods. Forty eight periods are represented by 360° which the whole circle subtends at the centre. You can determine the angle which will represent the periods of the different subjects.

$$\text{Angle at the centre of the sector representing Hindi} = \frac{360}{48} \times 9$$
$$= 67.5°$$

Similarly you can find the others.

Pie diagram showing the time allotted to different subjects in a week is drawn below :

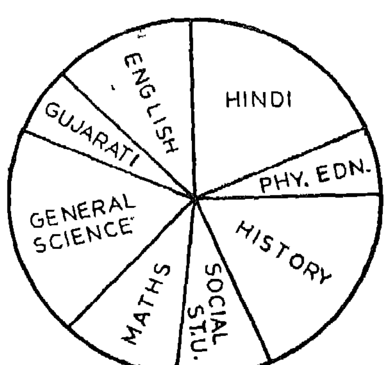



Fig 3.8

At the beginning of teaching pie diagram, the teacher may consider the time allotted by a student for his different activities (i.e. playing, studying and sleeping) during a day to be represented by a pie diagram.

---

Q. Where do you see a pie diagram ?

---

## Frequency Graphs

You know that a given set of data can be grouped and represented in the form of the frequency distribution table. We can also represent this frequency distribution graphically. These graphs give a better picture than their representation in tabular form. Three types of graphs can be used to represent the frequency distribution :

(1) The Histogram
(2) The frequency polygon
(3) The frequency curve

## Histogram

Look at a frequency table on the left side of this page. This is presented in the form of Histogram on the right side of this page.

| *Class interval* | *Frequency* |
|---|---|
| 120-130 | 4 |
| 130-140 | 5 |
| 140-150 | 13 |
| 150-160 | 20 |
| 160-170 | 8 |
| 170-180 | 6 |

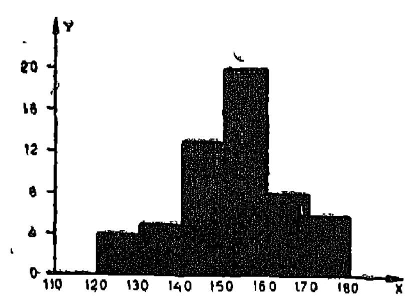



Fig 3 9

You can see that data are presented as a series of rectangles. Class intervals are plotted along the x-axis side by side. Take a class interval of 120-130 and draw on it a rectangle with the frequency '4' as its height. Take the second class interval 130-140 and draw a rectangle with the frequency 5 as its height adjacent to the first rectangle. You have seen that the frequencies are plotted along the y-axis. There are as many rectangles as there are classes. The height of each rectangle represents the corresponding frequency of that class. This gives a comprehensive

picture of the frequency distribution. The set of rectangles so drawn is called a histogram.

---

Q.1. What is the difference between a bar diagram and a histogram?

Q.2. Why do we start counting along the x-axis in a histogram not necessarily from zero?

---

### Frequency Polygon

Look at the histogram

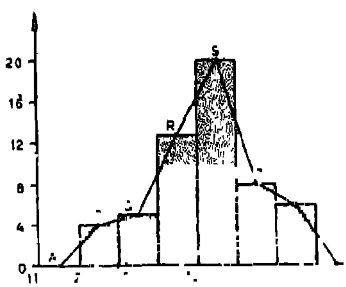



Fig 3 10

P,Q,R,S,T and U are the mid points of the tops of rectangles. The mid point of the top of a rectangle is joined by the mid points of the top of adjacent rectangles by straight lines. This is done on the presumption that the frequencies in a class are evenly distributed throughout the class and the mid point is representative of that class. To complete the polygon, the mid point **P** of the top of the first rectangle and the mid point U of the top of the last rectangle are connected to the point A and V respectively on the x-axis as shown in the figure. A and V are the mid points of the additional class-intervals introduced. You can see that the area of polygon is approximately equal to the area of histogram.

### Frequency Curve

You can see that frequency polygon is not a smooth curve. We can draw a smooth curve by joining various points of the polygon as below·

You can see that the area included between the smooth curve and x-axis is approximately equal to the area included between polygon and x-axis. Thus smoothening gives regular and continuous curve without disturbing the representation of the facts. It tries to eliminate all

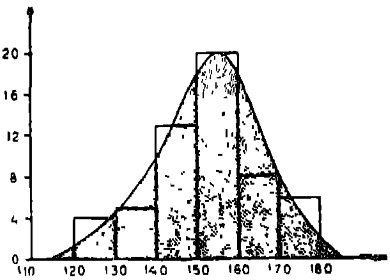



Fig 3.11

accidental variations that might creep in the data. In fact the width of the class interval is reduced indefinitely. We can interpolate from this curve the frequency o any intermediate size of the item. You can see frequency of 168 is ...8

---

Q. Construct a histogram, frequency polygon and frequency curve for the following data :

**Wages of workers in a Factory**

| *Wages in Rs. per day* | *No. of workers* |
|---|---|
| 10-20 | 3 |
| 20-30 | 10 |
| 30-40 | 14 |
| 40-50 | 24 |
| 50-60 | 17 |
| 60-70 | 14 |
| 70-80 | 3 |

---

Find out the number of persons getting wages Rs. 54/- per day from frequency curve.

---

**Ogive**

Look at the following frequency distribution table :

| *Class intervals* | *Frequencies* | *Cumulative frequencies* | |
|---|---|---|---|
| 1- 5 | 3 | | 3 |
| 6-10 | 4 | 3 +4 | = 7 |
| 11-15 | 6 | 7 +6 | =13 |
| 16-20 | 1 | 13+1 | =14 |
| 21-25 | 5 | 14+5 | =19 |
| 26-30 | 4 | 19+4 | =23 |
| 31-35 | 2 | 23+2 | =25 |
| 36-40 | 2 | 25+2 | =27 |
| 41-45 | 2 | 27+2 | =29 |
| 46-50 | 1 | 29+1 | =30 |
| | 30 | | |

By seeing this table a student can find the cumulative frequencies of different class-intervals, but it will be difficult for him to define or explain the meaning of cumulative frequency. Can we define the cumulative frequency of a class-interval as the total frequency less than the upper limit in each class ?

We can draw a graph to represent the relationship between the cumulative frequencies and the upper limits of the class-intervals. For example. Look at the graph drawn below:

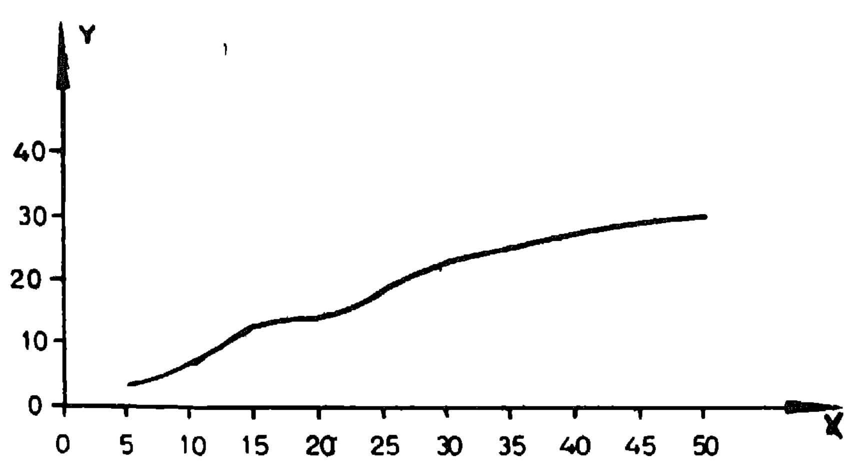



Fig 3.12

Here cumulative frequencies are plotted along y-axis, and the upper limit of class-intervals are plotted along x-axis.

---

Q Find out the following points in the above cumulative frequency graph :

(*a*) Where does the middle of the frequency lie ?

(*b*) Where does the quarter of the frequency lie ?

(*c*) Where does the three quarter of the frequency lie ?

---

**Teaching Strategies**

The main aim of teaching statistics at secondary level is that the children should be statistically literate. By reading this unit the student can understand statistical terminology. In the beginning of the lesson we must present statistical informations from the newspapers, magazines and other sources to show the necessity of learning statistics. Before teaching classification and tabulation, allow the students to read bar, histogram and pie diagram. You can prepare these diagrams relating to the different informations of your school. These can be :

(*i*) A bar diagram showing the total enrolment of the last five or six years.

(*ii*) A bar diagram showing the enrolment of students in different classes of the present year

(*iii*) A bar diagram showing the number of students appearing in Board Examinations during the last five years alongwith the results.

(*iv*) Pie diagram representing the area of campus used for classrooms. labs, office, agriculture farm, playground etc

(*v*) Picture diagram showing the number of boys and girls in different years etc.

(*vi*) Pie diagram showing the allotment of school hours for different activities.

(*vii*) A histogram showing the age of students in the school.

Similarly you can prepare various charts. Let the student learn to read these charts. Prepare a frequency distribution table by asking information from the students on the basis of the given histogram.

You can now pose the following problem before students . "If we want to prepare a histogram to show the heights of students of your class, how will we prepare it ?"

You can now write the heights of students of the class in *cm* on the blackboard. Ask the students : "If we want to prepare a histogram, how

can we groups them ?" Students will suggest some groups. By questioning and cross questioning clarify the following points :

(i) To form class intervals, locate the smallest and the largest datum.

(ii) Round up the smallest and the largest datum, i.e., to the nearest multiples of 5 or 10

(iii) Find the difference between the smallest and the largest data and divide it into different groups.

(iv) Length of interval should be the same.

Some of the cross questions can be :

(1) Why do we start class interval from 120 cms when the least height is 122 cms (say) ?

(2) Why do we not start class interval from 110 cms, when the least height is 122 cms (say) ?

(3) Why do we not always start from 0 ?

After deciding class-interval let them write the data in ascending order, then students will realise that it is laborious and time consuming. At this instance teach them how to mark tally to prepare frequency distribution. Students may be encouraged to think which of the two (frequency distribution table and diagrammatic representation of data) can be used more easily to get a result or inference. Give the student enough practice to represent the data diagrammatically.

While teaching pie diagram, let the student find out the formula for the angle at the centre from the given example and come to the conclusion.

$$\text{Angle at the Centre} = \frac{\text{Part represented} \times 360^\circ}{\text{Whole}}$$

**Evaluation**

Some of the instructional objectives of this unit are given below :

(1) Students will be able to prepare a frequency distribution table from the given raw data.

(2) Students will be able to draw a histogram and frequency curve for a given frequency distribution.

(3) Students will be able to represent certain information by pie diagram and bar diagram.

(4) Students will be able to draw Ogives for a given frequency distribution

**Specimen Test Items**

(1) Draw a bar diagram for the data given below :
Number of schools (Primary, Middle, Secondary, Higher

| | |
|---|---|
| Secondary P.U.C., Intermediate) | |
| Andhra Pradesh | 4,708 |
| Assam | 1,463 |
| J&K | 849 |
| Rajasthan | 3,413 |

(2) The following table gives the birth-rates and the death-rates per thousand of some countries. Represent them by a sub-divided bar diagram.

| *Country* | *Birth-rate* | *Death-rate* |
|---|---|---|
| India | 33 | 24 |
| Japan | 32 | 19 |
| Germany | 16 | 10 |
| Egypt | 44 | 24 |
| Australia | 20 | 9 |
| Newzealand | 18 | 8 |
| France | 21 | 16 |
| Russia | 38 | 16 |
| Pakistan | 16 | 11 |

(3) Draw a percentage bar diagram for birth-rates and death-rates of Q 2.

(4) The following table gives details of expenditure of a family. Represent them by pie diagram.

| *Items of expenditure* | *Amount* |
|---|---|
| Food | 250.00 |
| Clothing | 50.00 |
| House rent | 85.00 |
| Education | 15.00 |
| Miscellaneous | 100.00 |
| Total | Rs. 500.00 |

(5) The following table gives the age distribution of widows in India. Draw a histogram and a frequency curve.

| *Age-group* | *Widows (in thousands)* |
|---|---|
| 0-10.5 | 136 |
| 10.5-20.5 | 718 |
| 20.5-30.5 | 2,457 |

| | |
|---|---|
| 30.5-40.5 | 4,848 |
| 40.5-50.5 | 6,480 |
| 50.5-60.5 | 5,909 |
| 60.5-70.5 | 3,744 |
| 70.5-80.5 | 1,958 |

(6) Form the frequency distribution table of the following data giving the index of 68 commodities in a certain year.
113, 114, 114, 115, 116, 116, 117, 117, 116, 129, 120, 121, 79, 81, 86, 86, 87, 89, 91, 94, 95, 96, 96, 96, 122, 123, 124, 125, 128, 119, 134, 106, 76, 113, 114, 114, 114, 115, 116, 117, 117, 118, 119, 120, 121, 106, 107, 108, 108, 109, 107, 110, 110, 111, 112, 113, 99, 100, 100, 101, 101, 101, 102, 104, 104, 104, 105, 106.

(7) Answer item nos. 7 and 8 with reference to this accompanying diagram :
The number of students obtaining marks between 10 to 30 is :
(A) 10
(B) 20
(C) 25
(D) 21

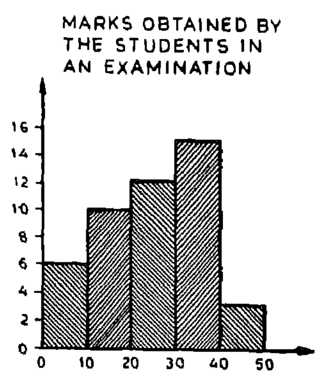



Fig. 3.13

(8) The maximum number of students have scored marks between :
(A) 10-30
(B) 20-40
(C) 30-50
(D) 0-20

(9) If the total population of the world is 120 million and the population of a country is 10 million, then the angle of the sector in pie diagram representing the population of this country will be :

(A) 30°
(B) 45°
(C) 60°
(D) 90°

(10) If the circle in the accompanying pie diagram represents the total expenditure as Rs. 600/- of a family then the expenditure on food is .
(A) Rs. 200/-
(B) Rs 150/-
(C) Rs. 100/-
(D) Rs 300/-

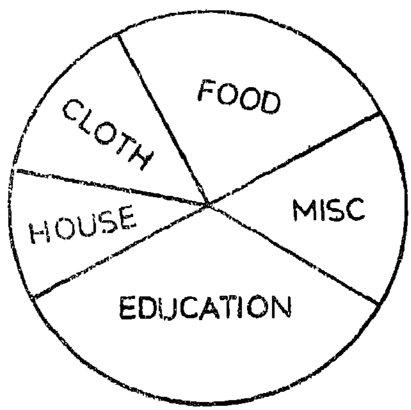



Fig. 3.14

(11) The circle is divided into three sectors. If the whole circle represents 24 hours, how many hours will be represented by a sector 30°?
(A) 2 Hours
(B) 4 Hours
(C) 6 Hours
(D) 8 Hours

(12) A whole circle represents 24 hours. What will be the angle subtended at the centre by a sector denoting 6 hours ?
(A) 30°
(B) 60°
(C) 90°
(D) 120°

**Assignments for Teachers**

(1) Describe briefly :
(*a*) Frequency distribution curves.

(*b*) Histogram.
(*c*) Cumulative frequency.

(2) What is the previous knowledge needed by a child to draw a histogram from a given frequency distribution ?

(3) Why do we call a circular representation a pie diagram ?

(4) Prepare test items to test the following behaviour ·
(*a*) Student will be able to calculate the angles of sectors to draw pie diagram.
(*b*) Student will be able to draw square-diagram for a given set of data

(5) Teaching of graphical representation must be from "Concrete to Abstract". Discuss.

(6) Write different types of data which can be collected from the class to teach classification.

(7) Draw squares for the following table which gives the production of wheat of the following countries in a certain year .

| *Countries* | *Production in quintals (1000,000)* |
|---|---|
| U.K. | 12 |
| India | 15 |
| Egypt | 11 |
| U.S A. | 230 |
| S. Africa | 3 |
| Canada | 108 |
| U.S.S.R. | 289 |
| Pakistan | 55 |

## MEAN AND STANDARD DEVIATION

### Introduction

We have discussed the representation of data by frequency distribution table. This is the first step in rendering a long series of observations comprehensible. But for practical purposes it is not enough, particularly when we want to compare two or more different series. For example :

(*i*) If two teachers examine the same set of answer books, they are liable to have different standards of marking Therefore it is necessary to have some method to compare their standards of marking.

(*ii*) If the same question paper is given to two classes, we need some method to compare the achievements of the students of two classes.

To compare two sets of data, we need number which is the best representative of a set for a certain purpose; one such number is known as mean

Consider the following scores of eleven boys in mathematics :

| I | II | III | IV | V | VI | VII | VIII | IX | X | XI |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 25 | 25 | 25 | 25 | 26 | 30 | 40 | 80 | 80 | 100 |

Now the question arises which is the best representative of these scores?

(*a*) Number $\frac{1+25+25+25+25+26+30+40+80+80+100}{11}$

i.e., 41.5 average score.

(*b*) 26, the middle mark.

(*c*) 25, the number occurring most frequently

All the three numbers—41.5, 26 and 25 are best representatives of the series. It is the purpose of the problem, which decides the one we have to consider. In this chapter we will study about the mean score 41.5.

**Content Covered in this Unit**

1. Mean :
   (*i*) Meaning.
   (*ii*) Methods of determining means for ungrouped data.
   (*iii*) Methods of determining means for grouped data.
   (*iv*) Short-cut methods.
   (*v*) Properties of Arithmetic mean.

**Meaning of the Mean**

The children have learnt to calculate average of given items in arithmetic. They can calculate the average by adding all the given items and dividing the sum by the number of items. For example :

Suppose there are ten students in a class. These students have amounts (in Rs.) in the bank as given in the following table .

| | | | | |
|---|---|---|---|---|
| 136 | 148 | 141 | 140 | 135 |
| 142 | 150 | 135 | 148 | 135 |

We are interested in finding out the average amount which we can obtain by adding all the amounts and dividing it by 10.

$$\text{Average amount} = \text{Rs. } \frac{1410}{10} = \text{Rs. } 141.$$

The arithmetical average is called the mean. The mean of the above amounts is Rs. 141/-

**To Calculate Mean of Ungrouped Data**

As given above the mean of the ungrouped data is obtained by adding all the data and dividing the sum by the number of data. If $x_1$, $x_2$, $x_3$, ...$x_n$ are data in a problem, then :

$$\text{Mean} = \frac{x_1+x_2+x_3+\ldots\ldots x_n}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i}{n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i$$

We generally represent mean by $\bar{x}$, x bar

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

For example : mean of 30,40 and 80 will be

$$= \frac{30+40+80}{3}$$

$$= \frac{150}{3}$$

$$= 50$$

---

Q.1. Express the sum $y_1+y_2+y_3+y_4+y_5$ useing '$\Sigma$' notation.

Q.2. Expand $\sum_{i=1}^{7} ai$

Q.3. Measure the lengths of five fingers of your right hand and find the mean length. Is the mean length equal to the length of fore-finger?

Q.4. Find $\sum_{n=1}^{5} n^2$

Q.5. Prove $\sum_{i=1}^{6} 4\, x_i = 4 \sum_{i=1}^{6} x_i$

Q 6 Prove . $\sum_{i=1}^{6} (x_i+y_i) = \sum_{i=1}^{6} x_i + \sum_{i=1}^{6} y_i$

---

**Method of Determining Mean of Grouped Data**

Supppose there are 30 students in a class. The individual height (in cms) of each student is as follows

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 120, | 120, | 120, | 124, | 124, | 124, | 124, | 125, | 126, | 126 |
| 130, | 130, | 130, | 130, | 130, | 130, | 130, | 122, | 123, | 121, |
| 132, | 132, | 132, | 135, | 135, | 135, | 135, | 135, | 135, | 135, |

We can represent the heights on a line by hollow circles. If two boys of the class have the same height, we will enter points on top of each other as shown in the figure below :

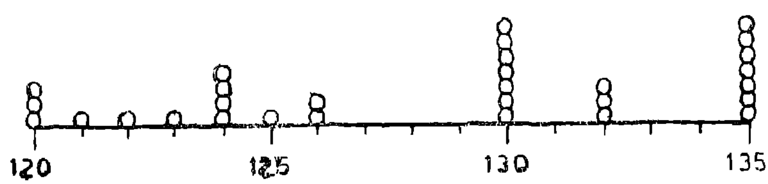



Fig 3.15

Here is a situation where we may have many students of same height. In this example we call the heights as variates and the number of students corresponding to a height as its *weight*. From this figure we can prepare the following data :

| Height (variate) | 120 | 121 | 122 | 123 | 124 | 125 | 126 | 130 | 132 | 135 |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of students (weight) | 3 | 1 | 1 | 1 | 4 | 1 | 2 | 7 | 3 | 7 |

$$\text{Mean height} = \frac{120\times3+121\times1+122\times1+123\times1+124\times4+125\times1+126\times2+130\times7+132\times3+135\times7}{30} \text{ cms.}$$

$$= \frac{360+121+122+123+496+125+252+910+396+945}{30} \text{ cms.}$$

$$= \frac{3850}{30} = 128.3 \text{ cms.}$$

In general let us consider the variates $x_1, x_2, x_3, \ldots x_k$ which are associated with the numbers called weights, $w_1, w_2, w_3 \ldots w_k$ respectively. Then the mean of such data is called weighted mean and is given by :

$$\bar{x} = \frac{x_1 w_1 + x_2 w_2 + x_3 w_3 + \ldots + x_k w_k}{w_1 + w_2 + w_3 \ldots + w_k}$$

$$= \frac{\sum_{i=1}^{k} x_i w_i}{\sum_{i=1}^{k} w_i}$$

If the variate $x_1$ has the frequency $f_1$, i.e., $x_1$ is repeated $f_1$ times and so on then

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \ldots + f_k x_k}{f_1 + f_2 + \ldots + f_k}$$

$$= \frac{\sum_{i=1}^{k} f_k x_k}{\sum_{i=1}^{k} f_k} = \frac{\sum_{i=1}^{k} f_k x_k}{n} = \frac{n}{1} \sum_{k=1}^{n} f_k x_k$$

Where $n = f_1 + f_2 + \ldots + f_k$

This formula can very easily be employed to find out the mean in the case of raw data as well as grouped data. The mean of a grouped frequency distribution can be obtained by the above formula after the classes have been replaced by their mid values, because we assume that the frequency in any class is centred at its middle point. Let us clear this point by considering the following examples :

**Example 1**

Calculate the mean of the following :

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Height in cms. | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 |
| Number of plants. | 1 | 4 | 5 | 7 | 11 | 10 | 6 | 4 | 2 |

Now here height is a variate. We can write it as $x$ and $f$ denotes the corresponding frequency (no. of plants). We prepare a table :

| *Height in cms xi* | *fi* | *fi xi* |
|---|---|---|
| 65 | 1 | 65 |
| 66 | 4 | 264 |
| 67 | 5 | 335 |
| 68 | 7 | 476 |
| 69 | 11 | 759 |
| 70 | 10 | 700 |
| 71 | 6 | 426 |
| 72 | 4 | 288 |
| 73 | 2 | 146 |
| TOTAL | $N = 50$ | $\Sigma fi\ xi = 3{,}459$ |

$$\text{Mean} = \frac{\Sigma fi\ xi}{N} = \frac{3459}{50} = 69.18$$

We can also find out the mean by directly substituting the values of $x_i$ and $f_i$ and so on in the formula :

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \ldots . x_k f_k}{f_1 + f_2 + \ldots \ldots f_k}$$

Q.1. $\frac{\Sigma fi\ xi}{n}$ can be derived from $\frac{\Sigma xi}{n}$ by using the fact that "Multiplication is repeated addition". Explain.

Q.2. What is the advantage of using the table to find $\Sigma fi\ xi$ over directly substituting the values of $f_1$ and $x_i$ in the formula for $\bar{x}$ ?

**Example 2**

Find the mean of the following distribution :

| *Monthly wages (in Rs.)* | *Number of workers* |
|---|---|
| 12.5-17.5 | 2 |
| 17.5-22.5 | 22 |
| 22.5-27.5 | 19 |
| 27.5-32.5 | 14 |
| 32.5-37.5 | 3 |

In this case $xi$ for each class interval is the class mark. The calculation of $\Sigma fi\, xi$ is shown in the following table :

| *Class* | *Mid-value* $xi$ | $fi$ | $fi\ xi$ |
|---|---|---|---|
| 12.5-17.5 | 15 | 2 | 30 |
| 17.5-22.5 | 20 | 22 | 440 |
| 22.5-27.5 | 25 | 19 | 475 |
| 27.5-32.5 | 30 | 14 | 420 |
| 32.5-37.5 | 35 | 3 | 105 |
| TOTAL | | $\Sigma fi = N = 60$ | $\Sigma fi\ xi = 1{,}470$ |

$$\text{Hence} = \frac{1}{N} \Sigma fi\ xi = \frac{1470}{60}\ 24\ 5$$

i.e., mean of grouped data = 24.50

Q.1. What is the assumption behind taking xi as variate?

Q.2. If the mean of marks obtained in a test for a group of 40 children is 50, and that with reference to the same test the mean for another group of 60 children is 25, what is the mean of the combined sample?

Q.3. Can you suggest some quick methods of multiplication and addition in the following cases ·

(*a*) 45 × 45 (*b*) 29+37

**Short-cut Method to Find out the Mean**

Look at the following table showing the age distribution of married females according to a sample census of 1941 in a State.

| *Age* | *No. of females* | *Age* | *No. of females* |
|---|---|---|---|
| 0-5 | 3 | 35-40 | 1292 |
| 5-10 | 31 | 40-45 | 963 |
| 10-15 | 410 | 45-50 | 763 |
| 15-20 | 1809 | 50-55 | 531 |
| 20-25 | 2446 | 55-60 | 317 |
| 25-30 | 2223 | 60-65 | 156 |
| 30-35 | 1723 | 65-70 | 59 |
| | | 70-75 | 37 |

We can determine the mean age of married females by using the formula .

$$\bar{x} = \frac{1}{n} \Sigma fi\, xi$$

This method gives accurate result, but involves tedius multiplication and addition. This is the reason for using some short-cut method to find the mean One of the short-cut methods is the method of **Assumed Mean**.

**Assumed Mean Method**

The most important fact to remember in calculating mean by this method is that we "guess" or "assume" a mean in the beginning and afterwards apply a correction to it in order to get the actual mean.

There is no set rule for assuming a mean. Any working mean may be taken, but as a convention, the middle value of the interval corresponding to the maximum frequency in the given distribution is to be taken as working or assumed mean. Once the question of A.M. (assumed mean) is settled, we would like to determine the correction which must be applied to the A.M. in order to get $\bar{x}$

In this method ·

(*i*) the deviation of each variable $xi$ from an assumed mean is first found out,

(*ii*) the deviations are then multiplied by their respective frequencies $fi$,

(*iii*) the total of the product $xi\, fi$ is calculated,

(*iv*) the total is divided by the sum of frequencies to find the correction mathematically

$$= A.M. + \frac{\Sigma fi\, di}{N}$$

where $di = xi - A.M.$

The above formula can be derived from

$$= \frac{\Sigma fi\, xi}{N}$$

Let $di = xi - A.M.$ or $xi = di + A.M.$

$$\therefore \bar{x} = \frac{fi\,(di + A.M.)}{N}$$

$$= \frac{\Sigma fi\, di}{N} + \frac{\Sigma fi\, A.M.}{N}$$

$$= \frac{\Sigma fi\, di}{N} + A.M. \frac{\Sigma fi}{N}$$

$$= \frac{\Sigma fi\, di}{N} + A.M. \left[ \because N = \Sigma fi \right]$$

In case of frequency distribution table having equal interval say of *range* "$h$", it is convenient to find out deviations in terms of range i.e. as multiples.

If $ui$ is the deviation of $xi$ from assumed mean in terms of class interval $h$, then

$$ui = \frac{xi - AM}{h}$$

or $$xi = AM + h\, ui$$

$$\Sigma\, fi\, xi = (AM)\, \Sigma fi + h\, \Sigma fi\, ui$$

or $$\frac{\Sigma fi\, xi}{N} = AM + h\, \frac{\Sigma fi\, ui}{N} [\because fi = N]$$

$$\boxed{\overline{X} = AM + h\, \frac{\Sigma\, fi\, ui}{N}}$$

It is easier to find out the mean by this formula. This is a short-cut method. Look at the following example :

**Example**

Calculate the mean of the following frequency distribution table :

| *Monthly wages (in Rs.)* | *No. of workers frequency* | *Monthly wages* | *No. of workers* |
|---|---|---|---|
| 12.5-17.5 | 2 | 37.5-42.5 | 4 |
| 17.5-22.5 | 22 | 42.5-47.5 | 6 |
| 22.5-27.5 | 19 | 47.5-52.5 | 1 |
| 27.5-32.5 | 14 | 52.5-57.5 | 1 |
| 32 5-37.5 | 3 | | |

**Solution**

In this distribution although the largest $f$ is in the interval of 17.5-22.5 yet it is not in the centre of the distribution. Therefore, it is not wise to take assumed mean 20. Let us take assumed mean in the interval of 22.5-27.5 and assume it to be 25. Hence the mid-value of this interval which is 25 is taken as Assumed Mean. The question of AM being settled, we like to determine correction.

Here a = 25 = A.M.

| *Mid value* $x_i$ | $f_i$ | $d_i = \frac{x_i - a}{h}$ | $f_i d_i$ |
|---|---|---|---|
| 15 | 2 | -2 | -4 |
| 20 | 22 | -1 | -22 |
| | | | -26 |
| 25 | 19 | 0 | 0 |
| 30 | 14 | 1 | 14 |
| 35 | 3 | 2 | 6 |
| 40 | 4 | 3 | 12 |
| 45 | 6 | 4 | 24 |
| 50 | 1 | 5 | 5 |
| 55 | 1 | 6 | 6 |
| | | | +67 |
| Total | N = 72 | $\Sigma f_i d_i$ | = 41 |

$$\text{Hence } \bar{X} = AM + h\frac{\Sigma f_i d_i}{N}$$

$$= 25 + 5.\ \frac{41}{72}$$

$$= 25 + \frac{205}{72}$$

$$= 25 + 2.85$$

$$= 27.85$$

*Note* : 1. The choice of A.M. = 25 is arbitrary.
2. $\bar{x} = 27.85$ is not 100% accurate.

**Evaluation**

(1) The age of 24 people in a certain income bracket is distributed as below. Find out the mean age.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Age :** | 29 | 33 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 45 | 47 |
| **Frequency :** | 1 | 1 | 3 | 4 | 2 | 3 | 2 | 2 | 2 | 3 | 1 |

(2) Find out the mean of the following frequency distribution.

| | | | | | | |
|---|---|---|---|---|---|---|
| **Class interval** | 81-83 | 78-80 | 72-74 | 69-71 | 66-68 | 63-65 |
| **Frequency** | 3 | 4 | 8 | 7 | 3 | 4 |

(3) Given the following frequency distribution, estimate the mean by short-cut method.

| *Class* | *Frequency* | *Class* | *Frequeney* |
|---|---|---|---|
| 171-175 | 4 | 146-150 | 19 |
| 166-170 | 8 | 141-145 | 17 |
| 161-165 | 14 | 136-140 | 11 |
| 156-160 | 22 | 131-135 | 3 |
| 151-155 | 27 | | |

For grouped data, the A.M. is given by

(*a*) $\bar{x} = \frac{\Sigma fx}{\Sigma f}$

(*b*) $\bar{x} = a + \frac{\Sigma fd}{\Sigma f}$

(*c*) $\bar{x} = a + h\ \frac{\Sigma fu}{N}$

(*d*) $\bar{X} = a + h.\ \frac{\Sigma fu}{\Sigma f}$

(*e*) All the above.

(4) Ramesh has 4 one-rupee notes, 10 ten-rupee notes, 5 five-rupee notes. The weighted mean is equal to :

(*a*) $\frac{4\times 1+10\times 10+5\times 5}{4+10+5}$

(*b*) $\frac{1+10+5}{4+10+5}$

(*c*) $\frac{1^2+5^2+10^2}{1+5+10}$

(*d*) $\frac{4+10+5}{4\times 1+10\times 10+5\times 5}$

**Assignments for Teachers**

(1) Any problem on mean involves a lot of multiplication and addition. A student commits errors in multiplication and addition while determining a mean. Due to this he feels that he is weak in the subject. What care should be taken to inculcate confidence in the pupil ?

(2) Give more situations from daily life where we use weighted mean.

(3) A tailor stitches ready-made clothes for 13-year-old girls. He will use

(a) mean measure of 13-year-old girls.

(b) middle measure of 13-year-old girls.

(c) The measure of 13-year-old girls with greatest frequency.

Give reasons for your answer

(4) You have 1000 marble balls of approximately equal dimensions. To measure the weight, we weigh only 20 marbles and determine the weight of 1000 marbles by calculation. Which measure we have used :

(a) Mean measure

(b) Middle measure

(c) Measure with greatest frequency.

## DISPERSION

### Introduction

It has been discussed that the comparison of two series can be done only if we obtain a single representative for each of them. This number is known as the measure of central tendency But considering the mean only is not sufficient for the purpose of comparison in many cases. This will be clear from the following problem.

The monthly income of two groups of people is as follows :

| *Income* | *Frequency* | |
|---|---|---|
| *(in Rs.)* | *Group A* | *Group B* |
| 0-100 | 12 | 2 |
| 100-200 | 14 | 6 |
| 200-300 | 27 | 23 |
| 300-400 | 18 | 38 |
| 400-500 | 3 | 25 |
| 500-600 | 6 | 2 |
| 600-700 | 15 | 1 |

The mean of both the groups is 350. Can we say two groups are compatible? If we see at the distribution, it is clear that the people in group B are clustered near the mean while people in group A are more dispersed. This leads us to devise a measure for the dispersion of the variates. In this we will discuss the method of finding dispersion.

**Content Covered in this Unit**

(1) Mean deviation
(2) Variance.
(3) Standard deviation.

**Deviations from the Mean**

Consider the marks of 11 students given below :

11, 10, 12, 21, 14, 3, 7, 25, 24, 32, 28.

The mean score is 17. We can find deviation of the marks of students from the mean by subtracting 17 from their score. Deviation of 11 from 17 is 11-17 i. e. $-6$. Similarly the deviations of eleven scores from the mean are :

$-6, -7, -5, +4, -3, -14, -10, 8, 7, 15, 11.$

If we add these deviations, the sum will always be zero. In order to get rid of negative sign, we consider the absolute values. Absolute deviations from the mean are :

6, 7, 5, 4, 3, 14, 10, 8, 7, 15, 11.

The sum of absolute deviations is 90. The mean deviation is obtained by dividing the sum of absolute deviations by the number of scores

$$\text{Mean deviation} = \frac{90}{11}$$

*Definition* : The mean of the absolute deviations from the mean is known as **mean deviation from the mean.**

If $x_1, x_2, \ldots\ldots, x_n$ are the given scores and $\bar{x}$ is their mean then :

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} | x_i - \bar{x} |}{n}$$

To find out the deviation of grouped data on a series let $x_1$ be repeated $f_1$ times, $x_2$ be repeated $f_2$ times and $x_3$ be repeated $f_3$ times and so on. And if their mean is $\bar{x}$, the mean of the absolute deviations from the mean is determined by using the formula :

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} f_i \, | x_i - \bar{x} |}{n}$$

Consider the following example :

**Example**

Find the mean deviations of the following data :

| $x_i$ | $f_i$ | $\mid x_i-\bar{x} \mid$ | $f_i \mid x_i-\bar{x} \mid$ |
|---|---|---|---|
| 43 | 2 | 3 | 6 |
| 42 | 1 | 2 | 2 |
| 41 | 1 | 1 | 1 |
| 39 | 3 | 1 | —3 |
| 38 | 3 | 2 | —6 |
| | N = 10 | | $\Sigma f_i \mid x_i-\bar{x} \mid = 18$ |

$$\text{Here } \bar{x} = \frac{43\times2+42\times1+41\times1+39\times3+38\times3}{10}$$

$$= \frac{400}{10} = 40$$

$$\text{Mean deviation} = \frac{\Sigma f_i \mid x_i-\bar{x} \mid}{n}$$

$$= \frac{18}{10} = 1.8$$

Q.1. Compute the mean deviation from the mean for the following data :

8, 10, 12, 14, 16, 18, 20, 22, 24, 26.

Q.2. Distribution of bar lengths in 100 bars is given below :

| | | | | | | |
|---|---|---|---|---|---|---|
| Length in cms | 15 | 16 | 17 | 18 | 19 | 20 |
| Frequency | 5 | 10 | 25 | 30 | 16 | 14 |

Assume the working mean as 18 cm. Compute the mean deviation.

If the class intervals and frequencies are given we can use the formula .

$$\text{Mean deviation} = \frac{f_i \mid x_i-\bar{x} \mid}{n}$$

Where xi is the class mark, i.e., the central value of the class interval. Study the following example :

**Example**

Calculate the mean deviation from the mean of the following ·

| *Class interval* | *Frequency* | *Mid-point of class interval* $xi$ | $\mid xi-\bar{x} \mid$ | $fi \mid xi-\bar{x} \mid$ |
|---|---|---|---|---|
| 2- 4 | 4 | 3 | 2 | 8 |
| 4- 6 | 3 | 5 | 0 | 0 |
| 6- 8 | 2 | 7 | 2 | 4 |
| 8-10 | 1 | 9 | 4 | 4 |

$$N = 10 \qquad \Sigma fi \mid xi-\bar{x} \mid = 16$$

$$\bar{X} = \frac{12+15+14+9}{10} = \frac{50}{10} = 5$$

$$\text{Mean deviation} = \frac{\Sigma fi \mid xi-\bar{x} \mid}{n} = \frac{16}{10} = 1.6$$

**Variance**

To get rid of negative values of deviations we have taken the absolute values of the deviations from the mean. Another method to get rid of negative values of deviations is to square the deviations from the mean and find the sum of the square of deviations.

Mean of the square of deviations is called the variance.

$$\text{Variance} = \frac{\Sigma (xi-\bar{x})^2}{n}$$

**Example 1**

Find the variance of the following :

Rs. 15, 20, 25, 30, 35.

**Solution** Mean $\bar{x} = 25$

Sum of squares of deviations from the mean :

$$= \Sigma (xi-\bar{x})^2$$
$$= (15-25)^2+(20-25)^2+(25-25)^2+(30-25)^2+(35-25)^2$$
$$= 100+25+0+25+100$$
$$= 250$$

$$\text{Variance} = \frac{\Sigma (xi - \bar{x})^2}{n}$$

$$= \frac{250}{5}$$

$$= 50.$$

Rs. 50.00 is the variance

**Example 2**

Compute the variance for the following :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Amount in Rs | 20 | 18 | 16 | 14 | 12 | 10 | 8 | 6 |
| Frequency | 2 | 4 | 9 | 18 | 27 | 25 | 14 | 1 |

**Solution**

$$\text{Mean} = \frac{20\times2+18\times4+16\times9+14\times18+12\times27+10\times25+8\times14+6\times1}{100}$$

$$= \frac{40+72+144+252+324+250+112+6}{100}$$

$$= \frac{1200}{100}$$

$$= 12.00$$

| *Amount in Rs.* | *Frequency* | $\mid x_i - x \mid$ | $(xi-\bar{x})^2$ | $fi(xi-\bar{x})^2$ |
|---|---|---|---|---|
| 20 | 2 | 8 | 64 | 128 |
| 18 | 4 | 6 | 36 | 144 |
| 16 | 9 | 4 | 16 | 144 |
| 14 | 18 | 2 | 4 | 72 |
| 12 | 27 | 0 | 0 | 00 |
| 10 | 25 | 2 | 4 | 100 |
| 8 | 14 | 4 | 16 | 224 |
| 6 | 1 | 6 | 36 | 36 |
| | N = 100 | | | $\Sigma fi(xi-\bar{x})^2 = 848$ |

Mean $= 12$

$$\text{Variance} = \frac{\Sigma fi(xi - x)^2}{n}$$

$$= \frac{848}{100}$$

$$= 8.48$$

$\frac{1}{N} \Sigma fi(xi-\bar{x})^2$ is known as variance or mean square deviation It is denoted by $\sigma^2$

Q.1. Complete the following table and compute

| *Class* | *Frequency* | $xi$ | $di = (xi-\bar{x})$ | $fidi^2$ |
|---|---|---|---|---|
| 0- 4 | 30 | | | |
| 5- 9 | 48 | | | |
| 10-14 | 12 | | | |
| 15-19 | 10 | | | |
| | N = 100 | | | |

Q.2. Compute $\sigma^2$ for the above problem taking 7 as working mean. Is the variance calculated approximately equal to the one calculated in Q 1. ?

Q.3. Compute $\sigma^2$ for Q 1 by using the formula

$$\sigma^2 = \frac{\Sigma fi\ (xi-a)^2}{N} - \left[ \frac{\Sigma fi\ (x_i-a)}{N} \right]^2$$

where $a$ is the assumed value of the mean (say $a$ =7)

Is the variance calculated here same as the one calculated in Q 1 ?

**Formula for $\sigma^2$**

Mean of a distribution is not always a whole number. So the deviations from the mean are expressed in the form of decimal fraction. Squaring of a decimal fraction and multiplying it with the corresponding frequency becomes time consuming and also difficult to calculate. In such cases, we assume a value for the mean (called working mean) and compute deviations from it. The working mean is usually a whole number and is so chosen that calculation of deviations from it is easy.

We can use the following formula to compute $\sigma^2$

$$\sigma^2 = \frac{1}{N} \Sigma fi\ (xi-a)^2 - \left[ \frac{\Sigma fi\ (xi-a)}{N} \right]^2$$

where $a$ is the assumed mean.

**Standard Deviation**

Positive square root of variance is called the standard deviation. It is denoted by S.D. If the variance of a distribution is 100, its S. D will be 10.

We can compare the two distributions on the basis of the means and standard deviations. The following are the scores of two batsmen A and B in a series of innings.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A. | 12 | 115 | 6 | 73 | 7 | 19 | 119 | 36 | 84 | 29 |
| B | 47 | 12 | 76 | 42 | 4 | 51 | 37 | 48 | 13 | 0 |

Mean of A = 50 and S.D. of A = 40.8
Mean of B = 33 and S.D. of B = 23.8

Looking at the mean and SD of A and B, we can say that :

(*i*) A is better run getter than B.

(*ii*) B is more consistent than A.

The distribution for which standard deviation is small is more clustered about the mean The standard deviation can thus be used for determining the scattering of data about the mean.

*Exercise* : Compute Mean and S D. for the following tables by choosing a suitable working mean.

| *A.* *Class intervals* | *Frequency* | *B.* *Class intervals* | *Frequency* |
|---|---|---|---|
| 0-20 | 0 | 0-20 | 6 |
| 20-40 | 4 | 20-40 | 6 |
| 40-60 | 10 | 40-60 | 6 |
| 60-80 | 5 | 60-80 | 6 |
| 80-100 | 5 | 80-100 | 6 |

| *Class intervals* | *Frequency* $f_i$ | *Class Marks* $x_i$ | *Deviation* $d_i=(x_i-a)$ | $f_i d_i$ | $f_i d_i^2$ |
|---|---|---|---|---|---|
| 0-20 | 0 | 10 | -40 | 0 | 00 |
| 20-40 | 4 | 30 | -20 | -80 | +1600 |
| 40-60 | 10 | 50 | 0 | 0 | 00 |
| 60-80 | 5 | 70 | 20 | 100 | 2000 |
| 80-100 | 5 | 90 | 40 | 200 | 8000 |
| N = 24 | | | | $\Sigma f_i d_i = 220$ | $\Sigma f_i d_i^2 = 11{,}600$ |

$$\text{Mean} = a + \frac{\Sigma fi\ di}{N} = 59.2$$

$$\text{S.D.} = \sqrt{\frac{1}{N}\Sigma fi\ di^2 - \left[\frac{\Sigma fi\ di}{N}\right]^2} = 19.9$$

| *B. Class interval* | *Frequency* $fi$ | *Class Mark* $xi$ | *Deviation* $di = (xi - a)$ | $fi\ di$ | $fi\ di^2$ |
|---|---|---|---|---|---|
| 0-20 | 6 | 10 | -40 | -240 | 9600 |
| 20-40 | 6 | 30 | -20 | -120 | 2400 |
| 40-60 | 6 | 50 | 0 | 0 | 0 |
| 60-80 | 6 | 70 | 20 | 120 | 2400 |
| 80-100 | 6 | 90 | 40 | 240 | 9600 |

$N = 36$ $\qquad\qquad \Sigma fi\ di = 0 \quad \Sigma fi\ di^2 = 24000$

$$\text{Mean} = a + \frac{\Sigma fi\ di}{N} = 50$$

$$\text{S.D.} = \sqrt{\frac{\Sigma fi\ di^2}{N} - \left(\frac{\Sigma fi\ di}{N}\right)^2} = 25.8$$

The measurers given above are not absolute measures of dispersion, and the resulting values in relation to two distributions cannot always be compared with any significance. To relate the measure of dispersion to its average and to convert it to percentage form, the S.D. is divided by Mean. This measure is known as coefficient of variation and is given by

$$CV = \frac{100\ \sigma}{Mean}$$

C.V. is generally used for comparison of variations or consistency of two or more quantities.

$$\text{For A. C.V} = \frac{100 \times 19.9}{59.2} = 33.6$$

$$\text{For B. C.V.} = \frac{100 \times 25.8}{50} = 51.6$$

This indicates that data (A) are more consistent than data (B).

We observe in the above example that the deviations are always multiples of class intervals. On dividing these deviations by the length of class intervals, we get small numbers.

Consider the following table

| *Class interval* | *Frequency* | *Class Mark xi* | $di = \frac{xi-50}{20}$ | *fi di* | *fi di*$^2$ |
|---|---|---|---|---|---|
| 0-20 | 0 | 10 | -2 | 0 | 0 |
| 20-40 | 4 | 30 | -1 | -4 | 4 |
| 40-60 | 10 | 50 | 0 | 0 | 0 |
| 60-80 | 5 | 70 | 1 | 5 | 5 |
| 80-100 | 5 | 90 | 2 | 10 | 20 |

$N = 24 \qquad \Sigma fi\, di = 11 \quad \Sigma fi\, di^2 = 29$

We can calculate the S.D. by using the following formula

$$\text{S.D} = \sqrt{\frac{\Sigma fi\, di^2}{N} - \left(\frac{\Sigma fi\, di}{N}\right)^2} \times k$$

Where $k$ is the length of interval and $di = \frac{xi - a}{k}$

Here :

$$\text{S.D.} = \sqrt{\frac{29}{24} - \left(\frac{11}{24}\right)^2} \times 20$$

$$= 20.$$

You can see that the calculation becomes easier in this case.

---

Q.1. If for two distributions ·

$M^1 = 30 \qquad S.D. = 20$

$M^2 = 40 \qquad S.D. = 30$

are known, which one is more consistent ?

Q.2. Calculate the S.D. from the following data :

| *Class interval* | *Frequency* |
|---|---|
| 3-4 | 3 |
| 4-5 | 7 |
| 6-7 | 22 |
| 7-8 | 60 |
| 8-9 | 85 |
| 9-10 | 8 |

Q.3. Prepare an exercise in which the student can get enough practice in using different formulae for computing S.D.

Q.4. If S.D. of a distribution is 20, what is its variation ?

---

**Teaching Strategies**

Students have already learnt to compute arithmetic mean in their previous classes. We can select two groups of students each consisting of seven or eight boys of the class and write their weights in kg. Let the students find out the arithmetic means of the two groups separately. Now we can ask them if $x_1\ x_2 \ldots\ . x_7$ are the weights of seven boys, what will be their mean weight ?

Generalize it to get

$$\text{Mean} = \frac{\sum_{i=1}^{n} xi}{\text{N}}$$

Give the students enough exercises for the use of "Sigma" notation. For example, consider the two groups of students and write their weights on a number line as below :

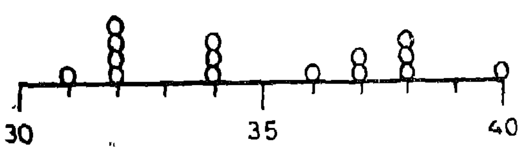



Fig 3.16

To find the weight of all the 15 boys, prepare the following table with the help of the students :

| $xi$ (*kg*) | *fi* (*frequency*) | *fi xi* |
|---|---|---|
| 31 | 1 | 31 |
| 32 | 4 | 128 |
| 34 | 3 | 102 |
| 36 | 1 | 36 |
| 37 | 2 | 74 |
| 38 | 3 | 114 |
| 40 | 1 | 40 |
| | N = 15 | $\Sigma fi\ xi = 525$ |

After preparing the table, students may be asked to find the total weight of 15 boys and compute the mean. Let them find out the formula to calculate mean in case of grouped data by using the formula :

$$\text{Mean} = \frac{\sum_{i=1}^{n} fi\, xi}{N}$$

The exercise can be extended to find mean weight of the whole class. Here frequency distribution table can be prepared by drawing tallies. If the frequency distribution table is given in the form of class interval, students can be asked to find the mean using the above formula. Here the following assumption should be made clear and explicit to the students.

**"The Frequency in any class is centred at its middle point".**

Here $xi$ will be the class-mark. Give one example in which class-marks and frequencies are large numbers and ask the students to find the mean by using the formula.

$$\text{Mean} = \frac{\sum_{i=1}^{n} fi\, xi}{N}$$

Then the students will realise that in such cases it is time consuming to calculate the mean.

To teach the short-cut method of finding mean, it is better to use deductive approach. Explain the following formula to the students to find out the mean.

$$\text{Mean} = a + \frac{\Sigma\, fi\, di}{N} \quad \text{where } di = xi - a$$

by taking an example on the blackboard. Now solve the above problem by this formula. Fill up the different columns of the table with the help of the students, so that students can appreciate the short-cut method. Explain to them clearly the method of choosing "Assumed mean". Again solve the same problem on the blackboard by using the formula.

$$\text{Mean} = a + h\, \frac{\Sigma fi\, di}{N} \quad \text{where } di = \frac{xi - a}{h}$$

Explain each term of the formula It will be better if a test is given at this instant wherein students have to determine the mean for the same distribution by using the three formulae for grouped data.

Look at the following sequence of teaching mean

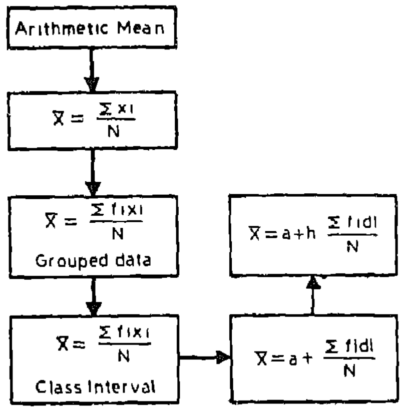



Fig 3 17

Give the following table of scores of two batsmen to the students and pose the questions and issues given below ,

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A. | 220, | 5, | 0, | 30, | 173, | 12, | 10, | 2, | 4, | 100 | = 556 |
| B. | 40, | 42, | 50, | 35, | 45, | 60, | 30, | 45, | 20, | 40 | = 407 |

Can you decide who is the best batsman? No doubt 'A' has a better average than B, but A is not consistent. How do we measure the consistency?

Now allow the students to examine the deviations of scores from the means for A and B. Now you can discuss mean variation, variance and standard deviation in that order. At the end you will use deductive approach to teach the formula for standard deviation Solve some exercises on the blackboard with the help of students to clarify the different notations used in the formulae.

$$\text{S.D.} = \sqrt{\frac{\Sigma fi\, di^2}{N} - \left(\frac{\Sigma fi\, di}{N}\right)^2} \text{ where } di = xi - a \qquad (1)$$

$$\text{S.D.} = \sqrt{\frac{\Sigma fi\, di^2}{N} - \left(\frac{\Sigma fi\, di}{N}\right)^2} \times h \text{ where } di = \frac{xi-a}{h} \qquad (2)$$

**Objectives**

Some of the instructional objectives of this lesson are given below :
Students will be able

(1) to compute means for a given raw data,

(2) to write the short-cut formula for the mean and give the meanings of different notations used in the formula.
(3) to complete the frequency distribution table to find out S.D.
(4) to write the relation between variance and S.D.
(5) to explain the use of S.D. with the help of examples.

**Specimen Test Items**

(1) Distribution of bar lengths in 100 bars is given below

| | | | | | | |
|---|---|---|---|---|---|---|
| Length in cms . | 15 | 16 | 17 | 18 | 19 | 20 |
| Frequency | 5 | 10 | 25 | 30 | 16 | 14 |

Assuming the working mean as 18 cm, compute :
(*i*) Variance and
(*ii*) Standard Deviation.
(2) The following are the marks obtained out of 100 by 10 students in mathematics.
Marks obtained . 0, 27, 80, 63, 25, 30, 35, 10, 58, 40. Calculate S.D.
(3) Calculate the S. D. from the following frequency distribution table ·

| | | | | | |
|---|---|---|---|---|---|
| Marks : | 1-10 | 10-20 | 20-30 | 30-40 | 40-50 |
| Frequency : | 4 | 8 | 20 | 8 | 2 |

(4) The heights of eleven boys of a college basketball team are 5′ 9″, 5′ 10″, 5′ 11″, 6′ 6″, 6′ 1″, 6′ 2″, 6′ 2″, 6′ 3″, 6′ 6″, 6′, 6′ 1″. Find the variance and standard deviation of the heights of these boys.
(5) From the prices of shares $x$ and $y$ given below, state which is more stable. Use standard deviation :

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | 55 | 54 | 52 | 53 | 56 | 58 | 52 | 50 | 51 | 49 |
| $Y$ | 108 | 107 | 105 | 105 | 106 | 107 | 104 | 103 | 104 | 101 |

(6) Find out the S.D. of the income of a certain person given in rupees for 12 months of a year :

139, 150, 151, 157, 158, 160, 161, 162, 162, 173, 175, 177.

**Assignments for Teachers**

(1) The following series present the marks obtained by 10 students in mathematics :

Marks obtained (out of 100) : 45, 76, 50, 30, 35, 33, 50, 60, 47, 78.

Find the variance.

(2) Find the S.D. of the following distribution :

| Marks | 0-5, | 5-10, | 10-15, | 15-20, | 20-25, | 25-30, | 30-35, | 35-40 |
|---|---|---|---|---|---|---|---|---|
| Frequency | 5 | 3 | 2 | 7 | 9 | 8 | 1 | 12 |

(3) (*a*) Define variance
(*b*) State the relationship between variance and S D
(4) Write the steps to compute Standard Deviation.
(5) The heights of eleven boys in a college hockey team are 5′ 9″, 5′ 10″, 5′ 11″, 6′, 4′ 5″, 4′ 7″, 4′ 9″ 5′, 5′ 3″, 5′ 9″, 6′. Find the S.D. of their heights
(6) From the following information regarding the marks obtained at college and the competitive examinations, find which group is more homogeneous ·

| *Marks* | *College examination No of students* | *Marks* | *Competitive examination No. of students* |
|---|---|---|---|
| 100-150 | 20 | 1200-1250 | 50 |
| 150-200 | 45 | 1250-1300 | 85 |
| 200-250 | 50 | 1300-1350 | 72 |
| 250-300 | 25 | 1350-1400 | 60 |
| 300-350 | 19 | 1400-1450 | 16 |

(7) Which of the following examples will we take in the beginning to teach variance and why?
(*a*) 23, 27, 30, 35, 27, 40.
(*b*) 5, 10, 15, 20, 25, 30, 35.
(*c*) 14, 20, 32, 40, 50, 60
(*d*) 5′ 9″, 5′ 10″, 5′ 11″, 6′, 4′ 5″, 4′ 3″, 5′, 5′ 3″, 5′ and 5′ 9″.
(8) Cite some examples to show the use of S D. in daily life.
(9) Collect some data relating to the students of your class to find mean and S.D.
(10) Give some illustrations where .
(*a*) Mean is zero.
(*b*) S.D. is zero.
(11) Is it possible to have S. D. zero while variance is not zero?

CHAPTER 4

# Elementary Trigonometry

## Introduction

The word "Trigonometry" means "measurement of triangles". The Greeks were among the first persons to discover trigonometry and use it for practical purposes. The circumference of the earth was measured by early men and their method depended on a very simple application of elementary trigonometry Even now trigonometry is widely used in surveying lands, etc.

But apart from using trigonometry for measurement purpose, trigonometry has been responsible for the development of the branches of mathematics. The persons who should be given credit for making trigonometry useful for the branches of mathematics are De Moivre and Euler. De Moivre is known to all college-going mathematics students for his De Moivre's theorem which states that

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

Euler will be always remembered in mathematical circles for the trigonometric relation

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

## Contents Covered in this Lesson

(1) Radian and degree measures of an angle.
(2) Trigonometrical functions.
(3) An application of trigonometric ratios.
(4) Trigonometrical formulae and identities.
(5) Graphs of elementary trigonometric functions

## Content-analysis and Development of Concepts

(*1*) *The concept of an angle*: Generally in the lower classes angle is defined as follows: "Two intersecting straight lines define an angle". But as you can see in the following figure, two intersecting lines determine not one but four angles—two pairs of equal angles, (possible) unequal angles being supplementary to each other Now this definition is quite restrictive in many senses. First, two intersecting straight lines do not

determine an angle uniquely. Secondly, the angle as is commonly measured by a protractor cannot have a measure greater than 360° or $2\pi$, i.e., the measure of an angle is bounded. Thirdly, an angle is not defined for negative numbers, i.e., an angle cannot have a negative number as its measure

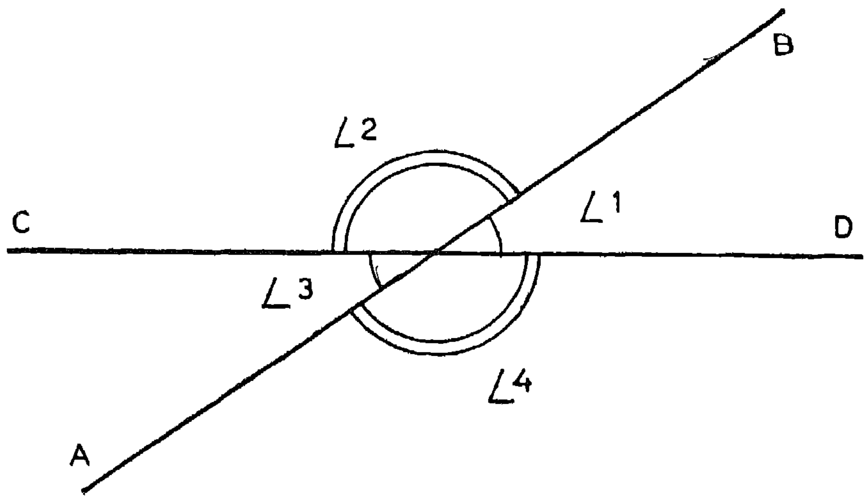
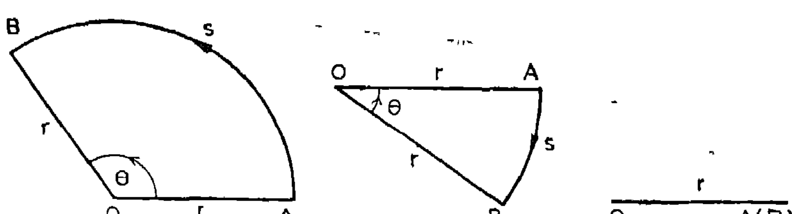



Fig 4.1

So we try to define an angle in a different way so that these restrictions are removed and our definition becomes more general.

Suppose that a ray from a point O in a plane rotates in the plane about O in a clockwise or an anti-clockwise sense from an initial position OA to a terminal position OB. The various cases arising thus are shown in the following figure.




(a) COUNTER CLOCK-WISE SENSE (Positive Angle)

(b) CLOCK-WISE SENSE (Negative Angle)

(c) NO ROTATION (Zero Angle)

Fig 4.2

Then the rotation is said to generate an angle AOB whose initial side is OA, terminal side is OB and vertex is O. To measure the amount of rotation which also gives the measure of the angle generated, suppose O is situated at the centre of a circle of radius r as shown in Fig 4 2 The directed distance may be measured along the arc AB which is described as the point A moves along the arc AB due to the rotation of the ray. Here, we suppose that $OA=OB=r$. We take s to be the positive or negative according as the rotation of the ray about 0 is in anti-clockwise or clockwise sense respectively. The circumference of the circle$=2\pi r$. Now if $s>0$ and $\frac{s}{2\pi r}=\frac{1}{360}$ we define that the measure of $<AOB=1$ degree, symbolically expressed as $1^\circ$ In other words, as a unit of measuring the angle, $1^\circ$ is equal to $\frac{1}{360}$ of a complete revolution of OA anti-clockwise. In degree measure, the value of any $\angle AOB$ as shown in Fig 4 2 is defined by

$$\theta = \frac{s}{2\pi r}(360) \text{ degrees} \quad \ldots \quad (1)$$

We agree that $\angle AOB$ is positive or has a positive measure if the rotation is anti-clockwise ($s>0$), and a negative value if the rotation is clockwise ($s<0$) If AOB consists of a complete anti-clockwise revolution (Note that OA comes to the position of OB in its terminal position as a sequel), then we have $s=2\pi r$ in (1) and $\theta=360^\circ$. We define that one minute $=1'=\frac{1}{60}$ or $1^\circ$ and one second$=1''=\frac{1}{60}$ of $1'$.

Now if the rotation is clockwise, s becomes negative and as a sequel the angle is defined to be negative or said to have a negative measure. It is easily seen that in this case too the relation (1) holds.

If there is no rotation, i.e., if $s=0$, it is said to be the zero angle or defined to have the measure zero. This definition of zero angle is easily found to be compatible with the relation (1).

If at any time any angle is shown without the sense of rotation of the ray, we will assume that the sense of rotation is anti-clockwise and the angle is positive. For example in the relation $\alpha+\beta+\gamma=180^\circ$ or $\angle A+\angle B+\angle C=180^\circ$, it is assumed that angles are positive. By this time it might have been clear that the symbol for an angle, say, $\alpha$ means the measure of the angle.

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**Q. 1. Trace the following angles and determine the corresponding sense of rotation in each case:**

**(a) 240° (b) 120° (c) −60° (d) 400° (e) −64°**

Q. 2 How will you explain to the students the difference between zero angle, a positive angle and a negative angle?

(2) *Radian Measure of an Angle*. We have degree measure or English measure or sexagesimal measure of an angle wherein we have $\frac{1}{360}$ of a revolution as the unit of measurement. We have already discussed this way of measuring an angle.

Another important method of measuring an angle is called Circular measure or Radian measure This method is described below

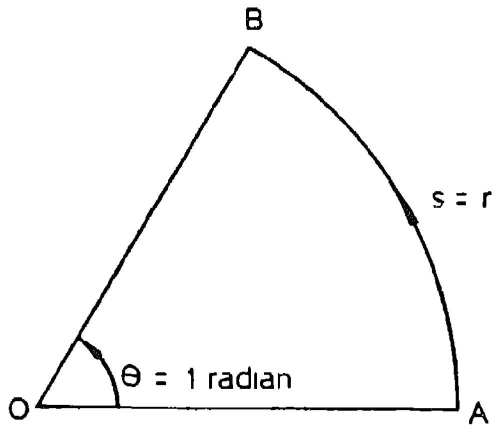


Fig. 4.3

Following the notations of section (1), we put $s=r$. Then we have an angle which is called one radian. 1 radian is taken as the standard unit of measuring the angle in many branches of mathematics, if we consider one complete revolution, $s=2\pi r$ which corresponds to $360°$ in degree measure.

Q. Demonstrate that a radian is little less than $60°$.

Let us consider a circle having radius r and centre O where A, B, C are points on its circumference such that $AB=r$, $AC=l$ and $\angle AOC=\theta$ radians. Then since in the case of a circle, its arcs are proportional to the angles subtended by them at the centre of the circle, we have:

$$\frac{\angle AOB}{AB} = \frac{\angle AOC}{AC}$$

$$\text{i.e.} \quad \frac{\text{I radian}}{r} = \frac{\theta \text{ radian}}{l}$$

$$\text{i.e. } \theta \text{ radians} = \frac{l}{r} \text{ radian}$$

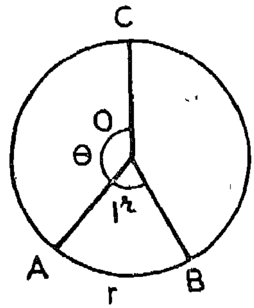


Fig 4.4

So if θ is measured in radians, then we have the formula

$\theta = \frac{l}{r}$. Customarily we write simply θ instead of θ radians. So when we write simply 10, this symbol means the angle whose measure is 10 radians

At this point it will be worthwhile to note how to convert radian to degree measure and vice versa. One revolution corresponds to

$$\frac{\odot^{ce}}{r} = \frac{2\pi r}{r} = 2\pi$$

radians and also to 360°.
So we have

$$2\pi \text{ radians} = 360^\circ$$

$$\text{i.e. } 1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = \text{about } 57^\circ$$

Conversely, we have $1^\circ = \frac{\pi}{180}$ radians.

Note that π is defined as $\frac{\odot^{ce}}{2\,(\text{radius})}$ which is an irrational number

A question arises, why we should have two methods of measuring an angle As an answer to this question, we should note that both the systems of measuring an angle have their respective advantages. In passing, it may be noted that radian (or circular) measure has been very helpful in the development of analytical trigonometry and so in the development of many branches of higher mathematics For example, in the famous Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, θ is in radian measure.

Another important point to be noted is that in both the systems, the units of measurement, i.e., a degree and a radian are independent of r, the radius of the reference circle

This is due to the geometrical theorem that arcs of different concentric circles subtending equal angles at the centre are proportional to the respective radii. Please refer to the following figure which is self-explanatory

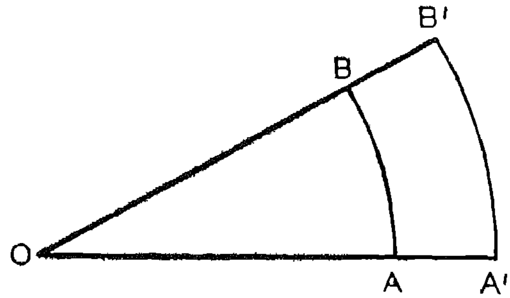


Fig. 4 5

In the figure, let OA=OB=r, OA'=OB'=r' OAA' and OBB' are two line segments. Then we have

$$\frac{\text{arc AB}}{r} = \frac{\text{arc A'B'}}{r'}$$

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Q.1. Express the following in degrees:

$2$, $3\pi$, $\frac{5\pi}{6}$ and $10$

Q.2. Express the following in radians.
$90°$, $400°$, $726°$, $527°$.

Q.3. Explain clearly why a degree and a radian are not dependent on r, the radius of the reference circle

Q.4 How will you explain to the students that a radian is a standard measure of measuring an angle, though we take help of the radius of a circle while defining a radian?

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*Trigonometric Function for $0° \leqslant \alpha \leqslant 90°$*

Initially we define trigonometric functions, for an angle $\alpha$ such that $0° \leqslant \alpha \leqslant 90°$. Then we extend our definitions of trigonometric functions for all real values of $\alpha$. In this connection the important role played by generalized definition of an angle may be noted.

We start defining $\sin \alpha$ and $\cos \alpha$ (read as "sine of $\alpha$" and "cosine of $\alpha$") for $0° < \alpha < 90°$ with the help of a right-angled triangle, $\alpha$ being an acute angle of the right-angled triangle Consider the following right-angled triangle ABC, where $\angle A = \alpha$ and $\angle B = 90°$

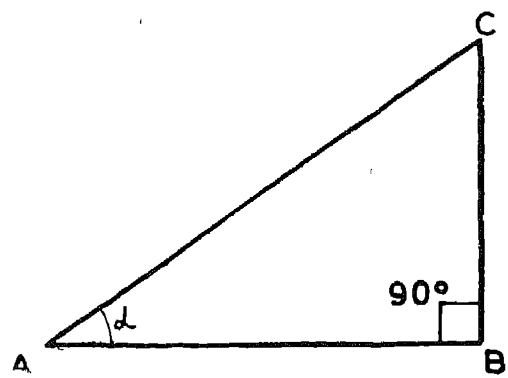


Fig 4.6

We define $\sin\alpha = \dfrac{BC}{AC}$ and $\cos\alpha = \dfrac{AB}{AC}$. Then we define other trigonometric functions of $\alpha$ e g tangent ($\tan\alpha$), cotangent ($\cot\alpha$), cosecant ($\operatorname{cosec}\alpha$) and secant ($\sec\alpha$) with the help of $\sin\alpha$ and $\cos\alpha$ as follows·

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{BC}{AB}, \quad \cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\tan\alpha} = \frac{AB}{BC}.$$

$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = \frac{AC}{BC} \text{ and } \sec\alpha = \frac{1}{\cos\alpha} = \frac{AC}{AB}$$

Then we define trigonometric functions for 0° and 90° with the help of limiting the process as follows

$$\sin 0^\circ = \lim_{BC \to 0} \frac{BC}{AC} = 0, \quad \cos 0^\circ \lim_{BC \to 0} \frac{AB}{AC} = 1$$

$$\tan 0^\circ = 1, \ \sec 0^\circ = 1$$

cot 0° and cosec 0° are not really defined though in common parlance each of them is taken to be equal to $\infty$. These notions will be made more clear while discussing the graphs of trigonometric functions. Readers can now discuss for themselves the trigonometric function of 90°

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Q.1 If in a right-angled triangle ABC, $AB = 3$, $BC = 4$, $AC = 5$ and $\angle BAC = \alpha$, then find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ and $\sec\alpha$

Q 2. Define cos 90°, sin 90°, cot 90° and cosec 90° as limits.

Q 3 How will you explain to the students that the definitions of trigonometric ratios for angles $0^\circ < \theta < 90^\circ$ with the use of a triangle naturally leads to the definitions of sin 0°, tan 0°, cot 90° etc as limits (wherever possible)?

Q 4. How will you explain to the students that tan 90° and cot 0° cannot be defined as limits?

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## Extension of Definitions of Trigonometric Functions to Any Angle

We define that an angle $\theta$ is in its standard position on a coordinate system if the vertex of $\theta$ is at the origin and the initial side of $\theta$ lies on the

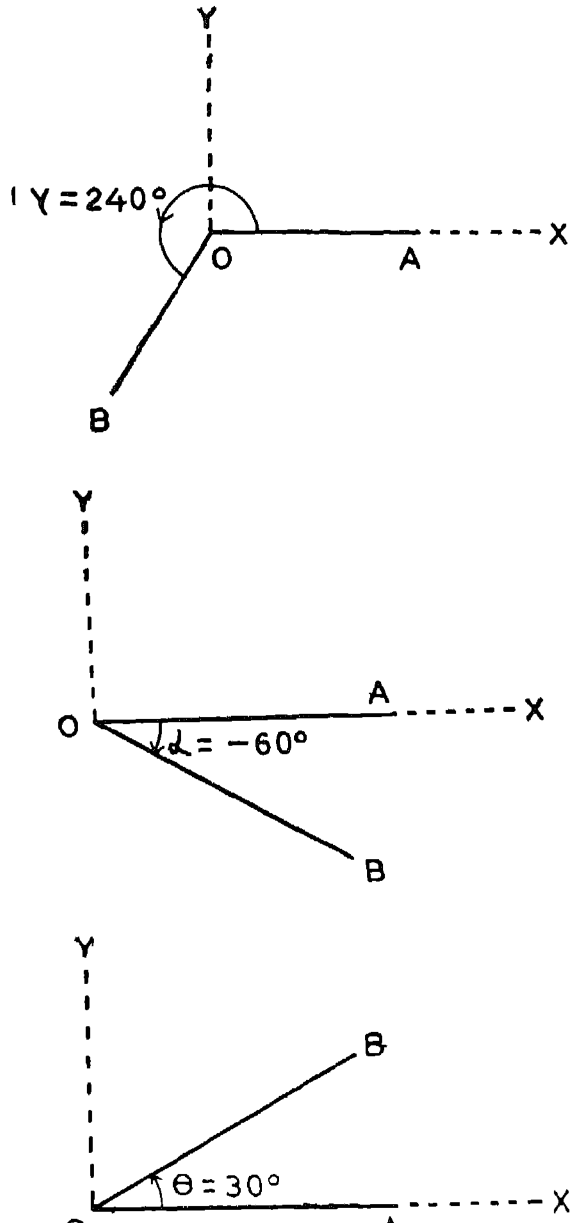


Fig. 4.7

positive part of the horizontal axis For example, see the following figures where $\gamma = 240°$, $\alpha = -60°$ and $\theta = 30°$ are shown to be in their standard positions.

To state that an angle $\theta$ is in a certain quadrant will mean that the terminal side of $\theta$ falls inside that quadrant when $\theta$ is in its standard position If the terminal side of $\theta$ falls on a coordinate axis, then $\theta$ is not said to be in any quadrant, but is called a quadrantal angle. For example, angles between $-180°$ and $-270°$ are in second quadrant, but $-180°$ and $-270°$ are measures of quadrantal angles.

Now let us extend the definitions of trigonometric functions for all real values of $\theta$.

*Definition.* Place angle $\theta$ in standard position on a coordinate system Choose any point P, not the origin, on the terminal side of $\theta$, let the coordinates and radius vector of P be (x, y) and r, respectively. Then the various trigonometric functions are defined as follows .

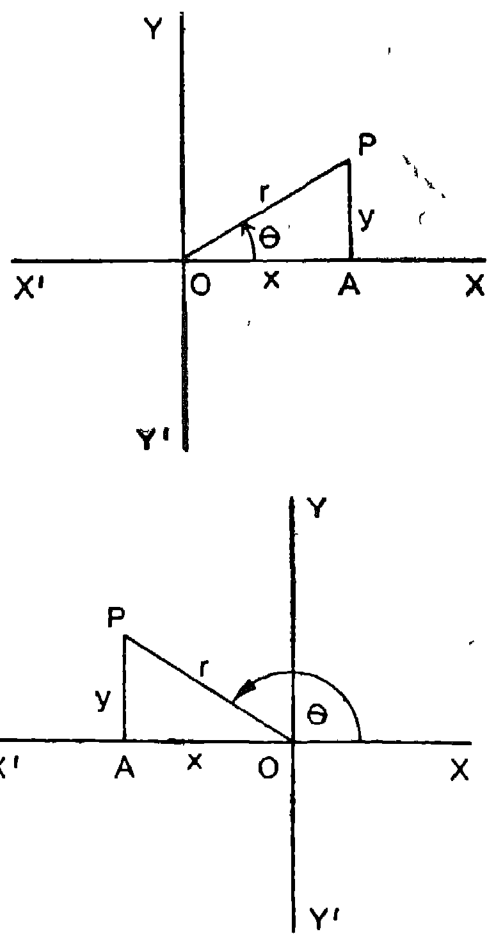


Fig. 4.8—I, II

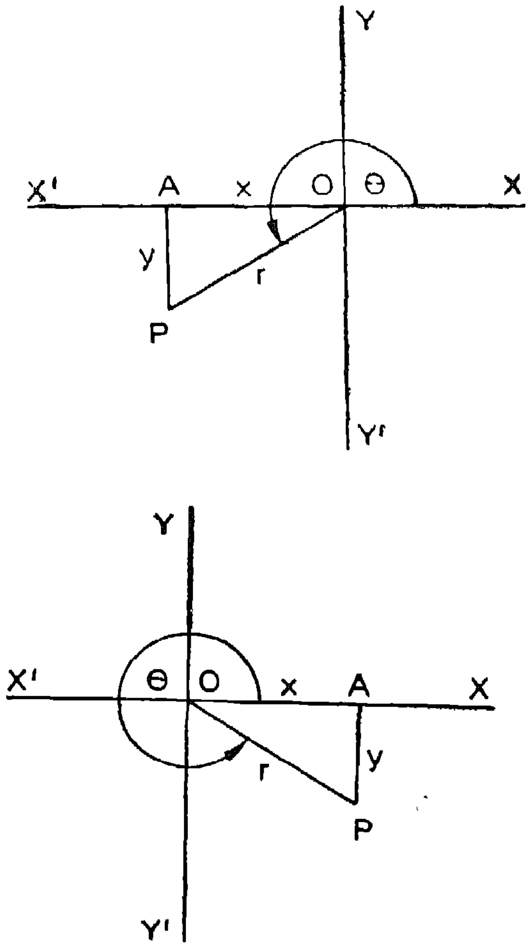


Fig 4.8—III, IV

$$\text{sine } \theta = \frac{\text{Ordinate of P}}{\text{radius vector of P}} = \frac{y}{r} = \sin\theta$$

$$\text{cosine } \theta = \frac{\text{Abscissa of P}}{\text{radius vector of P}} = \frac{x}{r} = \cos\theta$$

$$\text{tangent } \theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} = \tan\theta$$

$$\text{cotangent } \theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y} = \cot\theta$$

$$\text{secant } \theta = \frac{1}{\cos\theta} = \frac{r}{x} = \sec\theta$$

$$\text{cosecant } \theta = \frac{1}{\sin\theta} = \frac{r}{y} = \text{cosec }\theta$$

Since our extension of trigonometric functions involves only the terminal side of $\theta$, if two angles are co-terminal, the trigonometric functions of these angles are equal. For example, since 45° and 405° are co-terminal, sin 45° = sin 405°.

The sign of a trigonometric function depends upon the quadrant of $\theta$. For example, if $\theta$ is in the third quadrant, then $x < 0$, $y < 0$ and $r > 0$ in our definition. Hence, if we indicate only signs,

$$\text{Sin } \theta = \frac{y}{r} = \frac{-}{+} = -$$

$$\text{Cos } \theta = \frac{x}{r} = \frac{-}{+} = -$$

$$\text{Tan } \theta = \frac{y}{x} = \frac{-}{-} = +$$

Also we should note that for all values of $\theta$, since $|x| < r$, and $|y| < r$, we have

$|\sin\theta| \leqslant 1$, $|\cos\theta| \leqslant 1$

$|\sec\theta| \geqslant 1$, and $|\text{cosec }\theta| \geqslant 1$

The following table is generally useful in noting the interconnections among the trigonometric functions.

| *Angle θ in degrees* | *sin θ* | *cos θ* | *tan θ* | *cot θ* | *sec θ* | *cosec θ* | *Angle θ in radians* |
|---|---|---|---|---|---|---|---|
| 0° | 0 | 1 | 0 | none | 1 | none | 0 |
| 30° | $\frac{1}{2}$ | $\sqrt{3}/2$ | $1/\sqrt{3}$ | $\sqrt{3}$ | $2/\sqrt{3}$ | 2 | $\pi/6$ |
| 45° | $1/\sqrt{2}$ | $1/\sqrt{2}$ | 1 | 1 | $\sqrt{2}$ | $\sqrt{2}$ | $\pi/4$ |
| 60° | $\sqrt{\frac{3}{2}}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $1/\sqrt{3}$ | 2 | $2/\sqrt{3}$ | $\pi/3$ |
| 90° | 1 | 0 | none | 0 | none | 1 | $\pi/2$ |

*Def.* Let $\theta$ be an angle in any quadrant, and consider $\theta$ in standard position on a coordinate system. Then the reference angle for $\theta$ is defined

to be the acute angle $\alpha$ between the terminal side of $\theta$ and the horizontal coordinate axis For example, the reference angle of 120° is 60°.

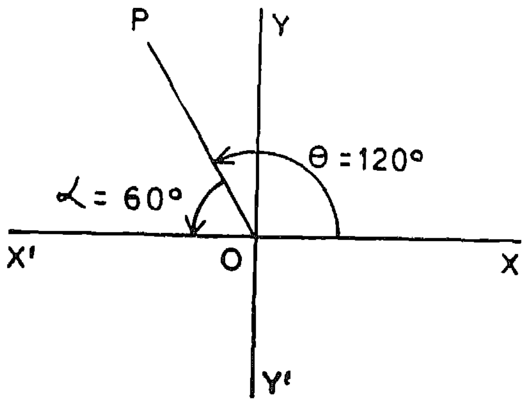


Fig. 4 9

It is now easily seen that any trigonometric function of $\theta = \pm$ (same triogonometric function of reference angle)

Q·1· Under given conditions, in which quadrant must $\theta$ lie?

(a) Sin $\theta < 0$ and tan $\theta > 0$

(b) Cosec $\theta > 0$ and cot $\theta < 0$

Q 2. Find the values of all trigonometric ratios when $\theta = 390°$.

Q.3. Why do we not have any value for Cosec 90° and cot 450°?

Q.4 How will you explain to the students that the definition of an angle as a measure of rotation and consequent definitions of trigonometric ratios are generalization of certain concepts?

**Infinite Values of Trigonometric Functions**

If $\theta \to 90°$ through values less than 90°, then tan $\theta$ increases indefinitely. This fact we denote by writing tan $90° = \infty$ But we should note that

$$\underset{\theta \to 90° + 0°}{\text{Lt.}} \tan \theta = -\infty$$

So also for sec 90° and cot 90° But it is customary to write that tan $90° = \infty$ ,cot $0° = \infty$ , cosec $0° = \infty$ and sec $90° = \infty$ .

Q. Discuss the behaviour of sec $\theta$ near $\theta = 90°$

**An Application of Trigonometric Ratios**

Trigonometric ratios have important applications in spherical trigonometry which is so useful to mariners and sailors. We will discuss here one

such application Before we come to the point, it is essential to explain some terms of spherical geometry and physical geography.

On a sphere, a great circle is one whose centre is the centre of the sphere, i.e , the plane of a great circle passes through the centre of the sphere Other circles on a sphere which are not great circles are called small circles

Meridians are great circles passing through the North and South poles. Equator is a great circle lying in a plane perpendicular to the line joining North and South poles.

The longitude of a place is the angle between the plane of its meridian and the plane of the Greenwich Meridian. The latitude of a place is its angular distance from the equator measured along its meridian It is North or South according as the place is located in the Northern or Southern hemisphere. The longitude of a place may vary from 180° E to 180° W, while its latitude varies from 0° to 90° (Northern or Southern hemisphere) The parallels of latitudes are not a great circle—they are small circles on the globe.

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Q 1. How many great circles can pass through North and South Poles?
(*a*) One, (*b*) two, (*c*) many

Q 2. How many small circles can pass through a given point?

Q.3. How many small circles are there, which are parallel to Equator and pass through a given point?

Q.4. Which is a Greenwich Meridian?

Q.5. Locate a small circle parallel to Equator at 30° s.

Q.6. Locate a point in the following figure, which is at 30° N latitude and 45° W longitude:

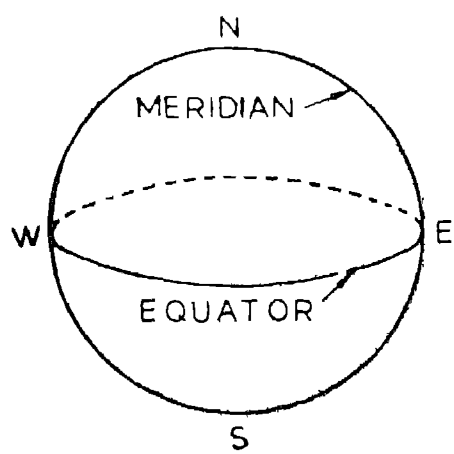


Fig. 4.10

Now we are ready to discuss the following problem which illustrates the use of trigonometric ratios in spherical geometry of the globe or earth.

**Problem 1**

Let P and Q be two points on the surface of the Earth on latitude $\alpha°$, their longitudes being $\beta_1°$, $\beta_2°$ East respectively ($\beta_1° > \beta_2°$). To determine the distance PQ along the latitude, take the radius of the Earth to be R miles.

Let the places P and Q be as shown in the figure.

By the question, $\angle POR = \alpha° = \angle OPO'$($\angle POR$, $\angle OPO'$ being alternate angles as O′ P and O R are parallel radii).

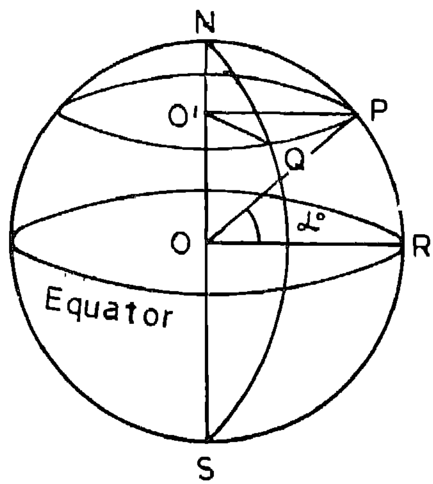


Fig. 4.11

O, O′ are the centres of the Equator and parallel of latitude.

Since $\angle OO'P = 90°$, we have $O'P = OP \cos\alpha° = R\cos\alpha°$

We may note that as $\alpha°$ increases, $\cos\alpha^c$ decreases and finally it reduces to 0 at the poles where latitude parallel reduces to a point.

Since P and Q are on longitudes $\beta_1°$ E and $\beta_2°$ E, $\angle ROP = (\beta_1° - \beta_2°)°$

$$\frac{\text{Arc PQ}}{R\cos\alpha°} = \frac{\pi}{180}(\beta_1 - \beta_2)$$

$$\text{Arc PQ} = \frac{\pi}{180}(\beta_1 - \beta_2) \times R\cos\alpha° \text{ miles.}$$

It may be of interest to note at this point that a nautical mile is defined as the length of the arc of a great circle subtending 1′ at the centre of the Earth. 1 nautical mile = 6080 ft. (1 knot = a speed of 1 nautical mile per hour = 1.15 miles per hour approximately).

Q.1 Taking the radius of the earth as 3960 miles, find the difference in the latitudes of two places one of which is 110 miles north of the other.

Q.2. Two places P and Q are on the same parallel of latitude and the meridians through these places intersect the equator at A and B. If the distance PQ along the latitude is 200 miles, and the distance AB along the equator is 250 miles, find the latitude of the two places.

**Graphs of Trigonometric Functions in Degree Measure**

If f (x) is a trigonometric function of x, then $f(x-360°) = f(x) = f(x+360°)$ where x is measured in degrees.

This can be easily seen from the following graph of sin x.

This fact is described summarily as follows All trigonometric functions are periodic functions with period 360°.

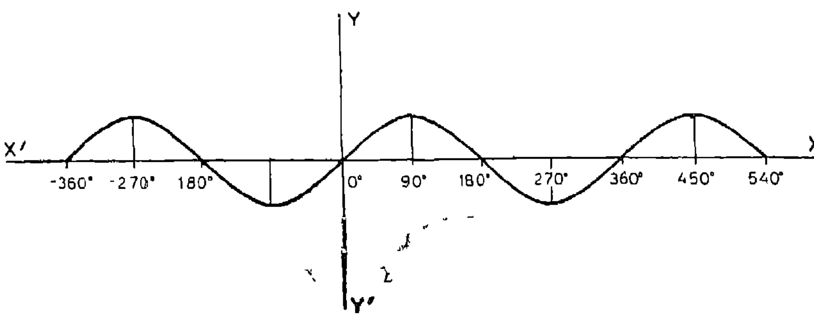


Fig. 4.12

Student-teachers, by certain manipulations of coordinate axes (say, shifting the origin along the x-axis by 180°) can find the different interconnections among the triogonometric functions. For example, sin x = cos (90°−x).

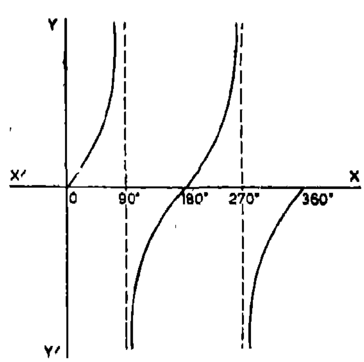


Fig. 4.13

Tan $x = \cot(90^\circ - x)$.

By drawing the graph, we can get a clear picture of the behaviour of a trigonometric function at certain points where it becomes $\infty$ (or, precisely not defined) For example, consider the graph of tan x in $0 \leqslant x \leqslant 180^\circ$.

We can clearly see from the graph that

$\lim_{x \to 90^\circ - 0^\circ} \tan x = +\infty$ and $\lim_{x \to 90^\circ + 0^\circ} \tan x = -\infty$

So, it is not mathematically correct that $\tan 90^\circ = \infty$ or $\lim_{x \to 90^\circ} \tan x = \infty$.

When we *customarily* say that $\tan 90^\circ = \infty$ we mean that $\lim_{x \to 90^\circ} \tan x = +\infty$.

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Q.1. Draw the graph of sec x when $-360^\circ \leqslant x \leqslant 360^\circ$.

Q.2. Analyse the statement $\operatorname{cosec} 0^\circ = \infty$ with the help of the graph of cosec x

Q.3. How will you clarify to the students the implications of the periodicity of trigonometric functions and the relation $\sec \theta = \operatorname{cosec}(90^\circ - \theta)$ by the graphs of sec and cosec?

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**Trigonometrical Formulae and Identities**

The basic formula in trigonometry, which is a reformulation of Pythagorean theorem, is as follows.

For every number s, $\sin^2 s + \cos^2 s = 1$.

Other formulae, with which we may start are the following:

$\sin(-s) = -\sin s$, $\cos(-s) = \cos s$, $\tan(-s) = -\tan s$ and $\cos(s_2 - s_1) = \cos s_1 \cos s_2 + \sin s_1 \sin s_2$ for any numbers s, $s_1$ and $s_2$.

From these basic formulae we can get the proofs of all other trigonometric formulae. For example, a proof of the formula $\cos(90^\circ - s) = \sin s$ may be the following We know that

$\cos(s_2 - s_1) = \cos s_1 . \cos s_2 + \sin s_1 \sin s_2$.

Now put $s_2 = 90^\circ$ and $s_1 = s$ in this and we get

$\cos(90^\circ - s) = \cos 90^\circ \cos s + \sin 90^\circ \sin s = \sin s$

It must be noted while teaching the proof of $\cos(s_2 - s_1)$, a general proof with the help of the unit circle should be given. It is better to avoid the multiplicity of particular cases in giving the proof. Afterwards other trigonometric formulae can be proved analytically.

From the graphs of trigonometric functions, the following types of reduction formulas can be immediately derived, e g.

$\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$

$\cos(-x) = \cos x$, $\tan(x \pm 90^\circ) = \tan x$

In proving an identity we should try to reduce a complicated expression to a simpler expression. For example, we prove the identity

$$\frac{\sec x}{\cot x + \tan x} = \sin x$$

$$\text{L.H.S} = \frac{\dfrac{1}{\cos x}}{\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}}$$

$$= \frac{\dfrac{1}{\cos x}}{\dfrac{\cos^2 x + \sin^2 x}{\sin x \cos x}}$$

$$= \frac{\dfrac{1}{\cos x}}{\dfrac{1}{\sin x \cos x}}$$

$$= \sin x$$

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Q.1. Prove the following identities by reducing the complicated expressions to simpler expressions

(1) $\dfrac{1-\tan^2 x}{1+\tan^2 x} = 1-2\sin^2 x$

(2) $\dfrac{1}{\operatorname{cosec} x-\cot x} - \dfrac{1}{\operatorname{cosec} x+\cot x} = \dfrac{2}{\tan x}$

(3) $1+\sin x = \dfrac{\cos x}{\sec x-\tan x}$

(4) Give an example to clarify to the students that not necessarily all trigonometric questions have solutions.

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**Teaching Methodology for a Selected Concept (The Concept of an Angle)**

Let us analyse the possible strategies for teaching the concept of an angle.

In the earlier classes the students learn that an angle is determined by two straight lines. So a teacher may find it difficult to reinforce the circular definition of an angle. So the students must be explained how the earlier definition hampers the development of the concept and how the extended definition helps us to define the trigonometric definitions for all real numbers Also, under the earlier definition the formula for sin (x+y) becomes meaningless in cases where x+y> 360°

Another difficulty is that the student fails to understand why we should have two units of measurement and why one unit is not sufficient To alleviate this study the *relative* strength and the limitations of the two units should be explained clearly by comparing the units. As an illustration, for practical purposes of measurement, definitely the degree unit

is preferable, whereas in analytic treatment of higher trigonometry the circular measure is preferable Even after finishing the concept of an angle, references should be made by the teacher to this point throughout the course in appropriate places.

The students also find it difficult to understand, how the units are independent of r, the radius of the circle. To remove this difficulty the teacher may correlate the concept of an angle with the following theorems of geometry.

(1) In a circle, arcs are proportional to their respective angles subtended by them at the centre

(2) If two arcs of two circles are proportional to their respective radii, then the arcs subtend equal angles at their respective centres.

**Generalisation of Concepts and Exceptions**

One should realise that the definition of an angle as a measure of rotation of a ray about its end-point is a generalisation of the concept of an angle as determined by two rays Consequently, definitions of trigonometric ratios of $\theta$ for real values of $\theta$ are generalisations of trigonometric ratios of $\theta$ for $\theta$ lying between 0 and 90°.

It should be emphasised while teaching that certain trigonometric ratios such as sin 0°, tan 90° and sec 90° stand as exceptions to some concepts in a certain sense. The exceptional characters of these trigonometric expressions become more clear if they are explained through graphs.

*Identities and equations*

While teaching trigonometric identities, it should be emphasised that not all trigonometric equations are identities. Trigonometric equations may be true for all values (as in identities) or for some values only (as in trigonometric equations). Some trigonometric equations may be contradictions, e.g., $\text{Sin}^2\,\theta+2=0$

**Teaching Strategies for Trigonometric-Identities**

We can classify Trigonometric-identities mainly into four classes on the basis of their method of proof.

*Class 1* Those identities, which can be proved by direct or indirect use of standard identities.

$$\sin^2\theta+\cos^2\theta = 1$$

$$1+\tan^2\theta = \sec^2\theta$$

$$1+\cot^2\theta = \operatorname{cosec}^2\theta$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta, \quad \frac{1}{\tan\theta} = \cot\theta$$

$$\frac{1}{\sec\theta} = \cos\theta$$

$$\frac{1}{\operatorname{cosec}\theta} = \sin\theta$$

For Example : Prove that

$$\frac{1}{1+\tan^2\theta} = \cos^2\theta$$

*Class II*. Those identities which can be proved by (*i*) changing the trigonometric ratios into sine and cosine and then (*ii*) using the standard identities, directly or indirectly.

For example $\quad \dfrac{1-\tan^2\theta}{\cos^2\theta} = \operatorname{cosec}^2\theta$

*Class III*. Those identities, which can be proved by (*i*) multiplying the numerator and denominator by a conjugate and then (*ii*) using the standard identities directly or indirectly.

For Example : Prove that

(1) $\quad \dfrac{1}{\sec\theta-\tan\theta} + \dfrac{1}{\sec\theta+\tan\theta} = \dfrac{2}{\cos\theta}$

(2) $\quad \sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta+\cot$

*Class IV :* Those identities which can be factorised :

For example

$$\cos^6\theta + \sin^6\theta = 1-3\sin^2\theta\cos^2\theta$$

In the beginning, we should give enough practice to our students in proving identities under class 1. It will help the students in learning the application of standard identities in proving other identities.

After proving enough identities under class I, we should give exercises in class II, class III and class IV At the end, we must give a miscellaneous exercise, which contains all classes of identities in random order.

**Evaluation**

Some of the instructional objectives of this lesson are given below. After studying this lesson the students will be able to :

(*a*) define an angle as a measure of rotation of a ray.

(*b*) define a zero angle, a positive angle and a negative angle.

(c) distinguish between a radian and a degree.
(d) define the trigonometric functions of an angle $\theta$ for all real values of $\theta$.
(e) prove the formula and identities involving trigonometric ratios.
(f) draw the graphs of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ for real values of $\theta$

**Evaluation Procedure for a Selected Concept or Topic**

Evaluation should be done on a continued basis during teaching and students can also be evaluated with the help of a test at the end of teaching a concept The test paper should contain graded items, starting with easier ones and proceeding gradually towards more difficult ones. The test should serve a two-fold purpose. First, with the help of test the teacher assesses the students and comes to know, how far his teaching strategy is successful and how far the students have been able to achieve the object. Secondly, the teacher has to carefully analyse the errors committed by the students. The following may be taken as a specimen test paper for the concept of an angle.

**Specimen Test Paper**

(1) Express the following as radians
(i) $30^\circ$ (ii) $45^\circ$ (iii) $-180^\circ$ (iv) $450^\circ$ (v) $405^\circ$
(2) Express the above quantities as multiples of radians.
(3) Change the given radian measure of an angle to degree measure·

(i) $\frac{x}{3}$ (ii) $-\frac{3x}{2}$ (iii) 4 (iv) $-\frac{7}{3}$ (v) $-\frac{9x}{4}$

(4) Express in radians ·

(i) $30^\circ\ 30'$ (ii) $120^\circ\ 15'$ (iii) $-405^\circ\ 15'$ (iv) $-360^\circ\ 45'$

(5) Through how many radians does the hour hand of a clock revolve in one hour 15 minutes?
(6) Through how many radians does the minute hand of a clock revolve in 15 minutes?
(7) On a circle, if an arc 50 cm long is intercepted by an angle of 3 radians at the centre, find the radius of the circle.
(8) On a circle of 20 cm in diameter, an arc subtends an angle of 2.4 radians at the centre. What is the length of the arc in centimetres?
(9) A railway line 1000 metre long is in the form of an arc of a circle having a radius of 1200 metres Find the angle in degrees through which the train will change its direction in going through this section of the railway line?

**Assignment for the Student-Teacher**

(1) Carefully study the content-analysis for the concept of trigonometric functions. Based on this content analysis what teaching strategies will you adopt to teach trigonometric functions ? Give reasons for your answer.
Also carefully prepare a test paper on the topic of trigonometric functions containing graded questions. This test paper should work as an achievement test as well as a diagnostic test.
Administer this test paper in a suitable class and analyse some of the typical errors committed by the students. Try to give reason, wherever possible, why students commit these errors.

(2) Where in trigonometry we use the following for concept-formation and argumentation:
   (a) generalisation of mathematical concept,
   (b) translation to mathematical symbols.

(3) Write behavioural objectives, corresponding learning activities and evaluation items for the following topics.
   (a) Definition of a radian and the formula $l = \frac{r}{\theta}$
   (b) Graphs of $\sin \theta$, $\cos \theta$ and $\tan \theta$ where $-180° \leqslant \theta \leqslant 540°$

(4) How will you explain the difference between trigonometrical identities and equations?

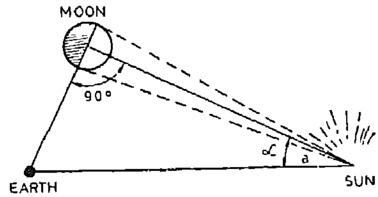
## HEIGHTS AND DISTANCES

**Introduction**

As has been remarked in the introduction of the previous unit, trigonometry was developed by the ancient intellectuals, because it had many uses for them. Lancelot Hogben in his "Mathematics for the Million" gives an interesting account, how Aristarchus used trigonometry for his purpose. Hogben's words are as follows

"Aristarchus attempted to make an estimate of the relative distances of the sun and moon from the earth and their relative sizes by observing the angle $\alpha$ in Fig 4 14 between the moon as seen in the early hours of morning, the sun, and the earth when exactly half the surface of the moon is visible, i.e. at half-noon. In the new dictionary language which we have now learned, the figure shows that

$$\sin \alpha = \frac{\text{moon's distance from the earth}}{\text{sun's distance from the earth}}$$




ARISTARCHUS (First Method)

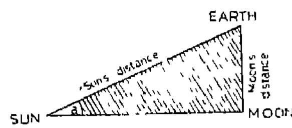



Fig. 4 14

With the crude instruments at his disposal he concluded that the angle was 3° He had no tables of angle ratios, so using a highly ingenious, though to us long-winded, application of Euclidean geometry, he showed that the sun's distance is between eighteen and twenty times the moon's distance from the earth. Not content with this method, he used a second one to check his result. The principle of this is not difficult to follow. What is fascinating about it is the daring resourcefulness of the performance when one considers the time at which it was made. It was the first great voyage of man's reasoning powers into the unchartered ocean of space.

**Content Covered in this Lesson**

(1) Definition of angles of elevation and depression
(2) Elementary problems of heights and distances (excluding three-dimensional cases).

**Development of Concepts: Definitions of Angles of Elevation and Depression**

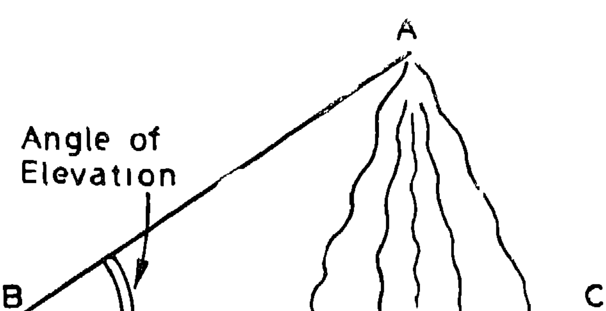



Fig 4.15

*Angle of Elevation* . Suppose A is the position of a building at the top of a hillock and B is the position of an observer in the valley. Let BC be the horizontal line through B in the vertical plane of the straight line AB Then ABC is defined as the Angle of Elevation of the object (building) at A as seen by the observer at B. (Refer to Fig. 4 15.)

*Angle of Depression* . Now let us take another situation (Fig. 4 16) Suppose that a man is positioned at A, the top of the hillock, and B is the position of an object in the valley.

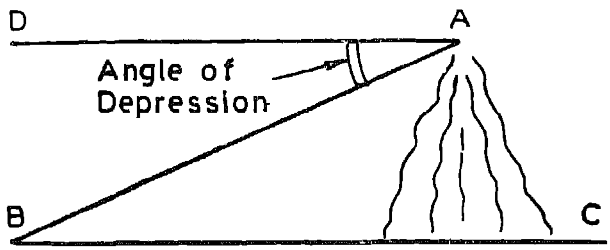



Fig. 4.16

Draw the line AD which is horizontal in the vertical plane of AB. Then $\angle$BAD is defined as the Angle of Depression of the object at B (in the valley) as seen by an observer stationed at A, the top of the hillock.

*Another Situation* · Let us now take the following situation. Suppose an observer A at the top of a tower is observing an aeroplane B flying in the sky and an object C situated on the ground in the vertical plane of the observer and the aeroplane. This situation is represented in the following figure : (Fig 4.17)

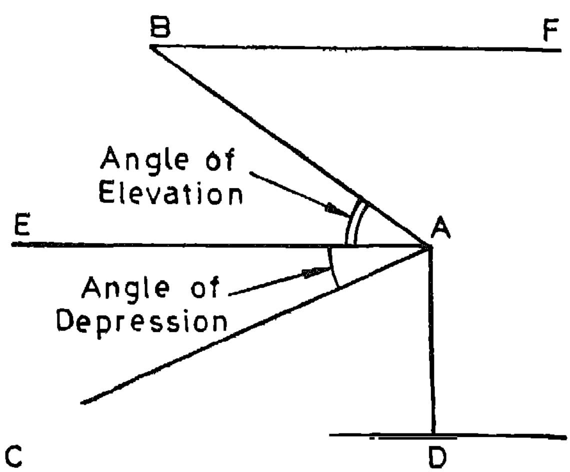



Fig. 4 17
(All the points in the figure are in the same vertical plane )

Here AD is the tower and CD, AE and BF are drawn parallel to the ground plane. Now we define $\angle EAB$ as the angle of elevation of the areoplane B with respect to the observer A and $\angle CAE$ as the angle of depression of the object C with reference to the observer A.

---

Q. In the following figure (Fig 4 18) the observer A at the top of a tower observes the object B above him and the object C on the ground Complete the figure to show the angles of elevation or depression (as the case may be) of B and C with respect to A.

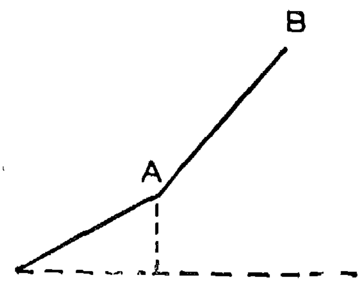



Fig. 4.18

(All the points in the figure are in the same vertical plane )

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**Concepts Involved in the Problems of Heights and Distances**

Concepts involved in the problems of heights and distances can be best explained by taking some typical and illustrative problems So let us discuss some illustrative problems

**Problem 1**

An aeroplane is 675 m directly above one end of a bridge. The angle of depression of the other end of the bridge from the plane is 60° Find the length of the bridge.

In Fig. 4.19 let AB be the bridge and C, the position of the aeroplane be such that $BC > AB$. Let $\angle CAB = 60°$ Now this figure gives a picturisation of Problem 1 which is a verbal problem.

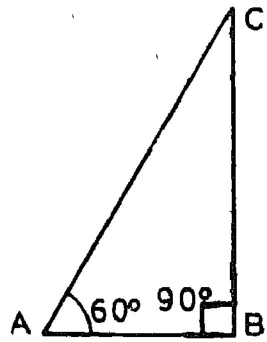


Fig 4.19

Now here $BC=675$ m

We get from trigonometry

$$\tan BAC = \frac{BC}{AB}$$

$$\text{or } \tan 60° = \frac{675}{AB}$$

$$\text{Therefore} \quad AB = \frac{675}{\tan 60°} = \frac{675}{\sqrt{3}} = \frac{675}{\sqrt{3}} \text{ m}$$

As it is quite clear, students are able to solve such problems provided students know the fundamentals of trigonometry and they are able to draw the correct geometrical figures characterising the facts of the problems

Now let us take a somewhat more complicated problem.

**Problem 2**

A vertical tower stands on a horizontal plane and is surmounted by a vertical flagstaff of height h. At a point on the plane, the angles of elevation of the bottom and the top of the flagstaff are respectively $\alpha$ and $\beta$. Prove that the height of the tower is

$$\frac{h \tan \alpha}{\tan \beta - \tan \alpha}$$

Fig. 4 20 picturises the facts of Problem 2. Here AB is the tower, BC is the flagstaff and D is the observer's position. Clearly, A,B,C are in the same straight line, $\angle ADB=\alpha$, $\angle ADC=\beta$ and $BC=h$ (given)

Now the problem is to express AB in terms of h, $\alpha$, $\beta$. We can introduce two unknowns x and y, where x is the length AB and $y=AD$

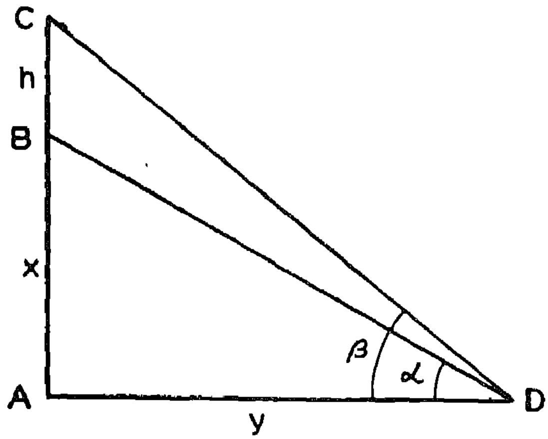


Fig. 4.20

It is now required to express x in terms of h, α, β. We form the following trigonometric equations with the help of the figure.

$$\frac{x}{y} = \tan \alpha \qquad \dots \quad (1)$$

$$\frac{x+h}{y} = \tan \beta \qquad \dots \quad (2)$$

We have thus two linear equations (1) and (2) in two unknowns x and y. So (1) and (2) determine x and y completely. But we are not required to find y in the problem. So it is better to eliminate y from (1) and (2) if this facilitates our calculation In this case, as can be seen from the equations, elimination of y from (1) and (2) makes the calculation very easy and gives immediately a simple equation in x only. So by division of the corresponding sides of (1) and (2) we get

$$\frac{x+h}{x} = \frac{\tan \beta}{\tan \alpha}$$

$$\text{or } (x+h) \tan \alpha = x \tan \beta$$

$$\text{or } x (\tan \beta - \tan \alpha) = h \tan \alpha$$

$$\text{or } x = \frac{h \tan \alpha}{\tan \beta - \tan \alpha}$$

---

Q. 1. A man observes that the angle of elevation of top of a cliff is 60°. The man is standing at a distance of 1000 metres Find the height of the cliff above the eyes of the man.

Q. 2. A man observes that the angle of elevation of the top of a vertical tower is 30° When he moves 30 metres in the direction of the tower, he finds the angle of elevation of the top of the tower to be 45° Find the height of the tower above the level of the eyes of the man.

Q. 3. (a) Devise a method to find the height of an inaccessible tower by means of observations made at distant points

(b) Analyse the cognitive processes by introspection that might have enabled you to solve the Q 3 (a).

---

**Teaching Strategies**

The unit on Heights and Distances in trigonometry involves a lot of problem-solving techniques, the problems being of the nature of word problems. In this unit, the teacher's objective should be to enable the students to solve the word problems for themselves. To be able to solve the problems in this unit, the students should have a thorough grasp of the fundamentals of elementary trigonometry and they should be able to draw correct geometrical figures for the problems

Generally, it has been found that students fail to draw correct geometrical figures because they fail to grasp the import of the language of the problems. So the first duty of the teacher should be to train the students in analysing the problems so that they may be able to draw correct figures. One way to enable the students to analyse the problems for themselves is to take the real life situations and compare these with the situations described in the problems. Such real life situations can be thought of as measuring the heights of electric poles and certain multi-storeyed buildings on the roadside or measuring the dimension of play-grounds. In analysing a problem the students should be specifically asked to identify the given elements and the element to be determined in the problem.

Next step in solving the problems of heights and distances is to write down the correct equations leading to the solution of the problems. This involves introduction of new unknowns which may help in writing down the correct trigonometric equations. For example, in problem 2 of content analysis section we introduce the unknown x and y, where y=AD and AB=x. It should be noted that the unknowns to be introduced in a problem should preferably denote some horizontal and vertical lengths, because such a choice of unknowns facilitates calculations. For example, in problem 2 of content analysis section we do not introduce unknowns to represent BD or CD which are neither vertical nor horizontal.

Generally, the teacher should give the students sufficient practice in drawing figures for the problems.

The number of equations should be equal to the number of the unknowns so that all the unknowns are completely determined in terms of given or known quantities Another way is to eliminate the unwanted unknowns (e.g., y in problem 2) from the equations to reduce the number of unknowns and correspondingly the number of equations. This alternative method is illustrated while discussing the solution of problem 2 in the section on content analysis.

Generally a class on problems on 'heights and distances', turns out to be a problem-solving session So the teacher should see that each student is given individual attention and his mistakes are rectified. For this the teacher should give home-assignments and correct the assignments regularly.

**The Use of Clinometer**

For practical experience students may be asked to prepare a teaching aid called clinometer, which can be used to find the heights of the objects which are situated above the observer. A description of clinometer and its use is given below .

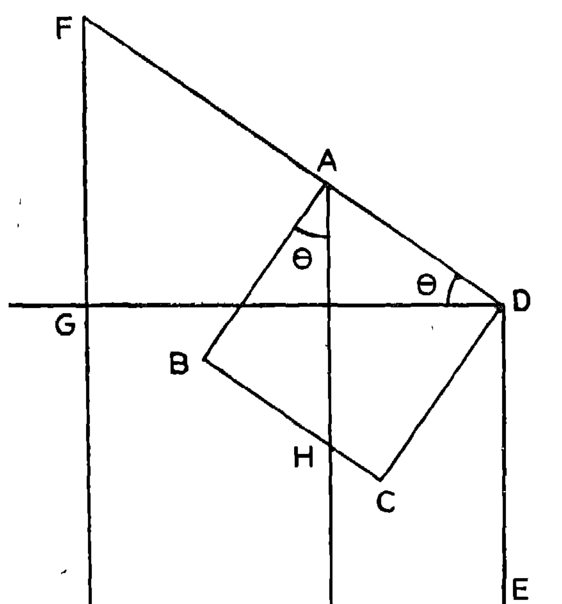


Fig. 4 21

Suppose DE is observer's height and D is the position of the observer's eyes. Let F be the object to be observed at D.

The square ABCD represents the clinometer which the observer is to use. In this clinometer the sides AB and BC are graduated *in the same scale* and along AD a thin tube (open at both ends) is fixed so that the observer can observe the object at F through this tube. Now, if $\theta$ is the angle of elevation of F with respect to D, then

$$\theta = \angle GDF = \angle BAH$$

$$\text{Then } \tan\theta = \tan\angle GDF = \tan\angle BAH$$

$$= \frac{BH}{AB}$$

which can be known by measuring AB and BH since sides AB and BC are graduated in the same scale.

Let us now suppose that DG = a and DE = h; and we have to find the height x of F above the ground level.

Since we have $\tan\theta = \dfrac{FG}{GD}$,

We have

$$\begin{aligned} x &= DE + FG \\ &= h + DG\tan\theta \\ &= h + a\tan\theta \end{aligned}$$

Now, h is known, a being a distance on the ground can be measured and the value of $\tan\theta$ is given by clinometer. This is how clinometer helps us in finding the height of an inaccessible object.

The use of clinometer in teaching makes the lesson interesting and instructive and, moreover, it is easy to prepare this teaching aid.

**Evaluation Techniques**

While evaluating the students, the teacher should take the following into consideration :

(1) Students should be given due credit if they are able to draw the correct figures, though they may not be able to write the correct equations.

(2) Next to drawing the correct figures, the important thing is that students write the correct equations (as far as possible there should be less number of equations introducing the minimal number of unknowns).

(3) The teacher should give due credit to the computational abilities of the students.

(4) In a unit test, the test paper should start with easy questions, gradually moving towards more complicated problems. A sample test paper is given to show, how a test paper can contain graded problems and a variety of concepts related to heights and distances.

**Objectives of the Lesson**

Some of the instructional objectives of this lesson are given below. After studying the lesson the students will be able to .

(a) define the angles of elevation and depression,

(b) analyse the simple problems as "heights and distances" by drawing correct figures and writing correct equations,

(c) solve problems on heights and distances when the situation of the problem can be represented by a plane figure, and

(d) use clinometer for finding the heights of inaccessible objects.

**Specimen Unit Test**

Let question Nos. 1,2,3 of the unit test be problems 1,2,3 of the content analysis section, in that order. Additional problems may be given as follows in the order given here after problem Nos. 1,2,3.

(4) A man on the top of a vertical lighthouse observes a boat coming directly towards it. If it takes 10 minutes for the angle of depression to change from 30° to 60°, how soon will it reach the lighthouse ?

(5) At a point midway between two towers on a horizontal plane the angles of elevation of their tops are 30° and 60° respectively. Show that one tower is three times as high as the other.

(6) From each of two stations east and west of each other and 1 km apart, the angle of elevation of a balloon is observed to be 45°. If the balloon bears N.W. and N.E. from the stations, respectively, how high is it ?

(7) In order to measure the height h, an object, the distance between two points, A and B, along a line through its base in a horizontal plane is measured and found to be l m long. The angles of elevation of the top of the object from A and B are found to be $\alpha$ and $\beta$ respectively, A being nearer the base. Show that the height is given by the formula

$$h = \frac{l}{\cot \beta - \cot \alpha},$$ if A and B are on the same side

and by $$h = \frac{l}{\cot \beta + \cot \alpha},$$ if A and B are on opposite sides of the base.

(8) The pilot in an aeroplane observes the angle of depression of a light directly below his line of flight to be 30° A minute later its angle of depression is 45°. If he is flying horizontally in a straight course at the rate of 90 km per hour, find (a) the altitude at which he is flying (b) his distance from the light at the first point of observation.

**Assignments for the Student-Teacher**

(1) Read carefully the problems (not worked out) given in the section on the content analysis and state how you will analyse the problems so that the students are able to draw the correct figures.

(2) Cite some real life situations (easily accessible to a typical school student of your area) to illustrate the methods and concepts involved in the solutions of the problems given in this unit.

(3) Solve the solved example of problem 2 of the section on content analysis without eliminating $y$ from the two equations.

(4) Read carefully all the exercises given in the chapter on Heights and Distances of a standard plane trigonometry textbook. Select from these exercises those ones which can be solved without the use of trigonometric or logarithm tables and grade the selected exercises according to difficulty level

(5) Explain how teaching of "heights and distances" enable the students to visualise the connection of mathematics with everyday life.

(6) Consider question No 8 of specimen unit test Analyse at what levels of mental domain the students will be working, if they are able to solve the problems *themselves*.

(7) Suppose you are doing problem No. 7, of specimen unit test. Write the instructional objectives, and corresponding teaching strategies Analyse the behavioural changes in the students also.

CHAPTER 5

# Geometry

## CONGRUENCE OF TRIANGLES

**Introduction**

When we compare two plane figures with reference to their shapes and sizes, there are three possibilities .

(1) The figures have neither the same shape nor the same size. For example, a triangle and a circle.
(2) The figures have the same shape but not the same size; for example, two circles of different radii. Such figures are called **Similar Figures.**
(3) The figures have the same shape and same size. For example, two circles of same radius. Such figures are called **Congruent Figures.**

It is easy to see when two circles have the same radius, they are congruent. Now the question arises when two triangles will be congruent. We will study about congruence of triangles in this lesson.

**Content Covered in this Lesson**

(1) Concept of congruence.
(2) Conditions for congruence .
    (a) angles
    (b) segments
    (c) triangles
    (d) right-angled triangles.
(3) Previous knowledge needed to prove that two triangles are congruent
(4) Application of congruence of triangles.

**(1) Concept of Congruence of Plane Figures**

Let us consider two triangles.

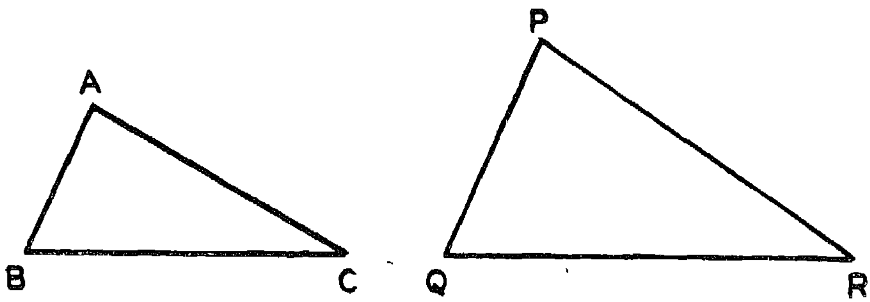


Fig. 5.1

If we place △ PQR over △ ABC, we can cover △ ABC but can we cover △ PQR by placing △ ABC over it ? Clearly no. Again look at the following triangles.

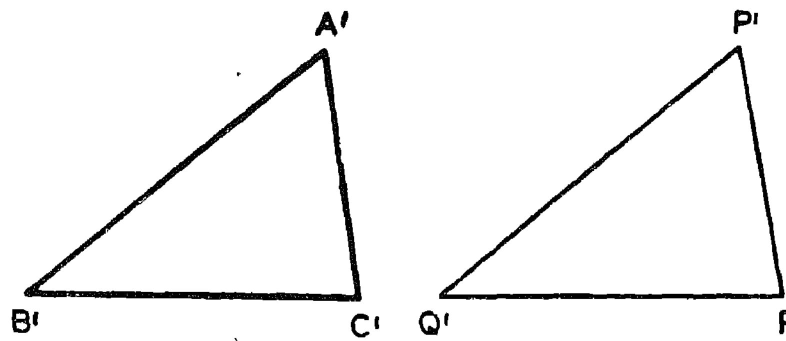


Fig. 5.2

In this case you can cover completely △ A'B'C' by putting △ P'Q'R' over it. We can also cover completely △ P'Q'R' by putting △ A'B'C' over it.

Two figures are said to be congruent if by placing one figure on the other they cover each other completely.

**(2) Conditions of Congruence of Figures**

(*a*) *Congruence of Angles*

Consider the following pair of angles :

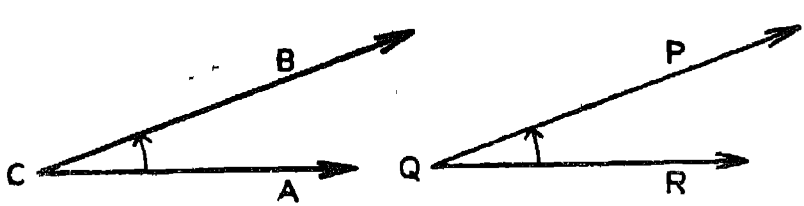


Fig. 5.3

If you place Q at C and $\overrightarrow{QR}$ along $\overrightarrow{CA}$, $\overrightarrow{QP}$ will fall along $\overrightarrow{CB}$. Thus $\angle PQR$ will cover $\angle BCA$ exactly. In this case we can say, the two angles are congruent

(*b*) *Congruence of Line Segments*

Look at the following line segments $\overline{AB}$ and $\overline{PQ}$


Fig 5.4

If P is placed on A and PQ runs along AB, then Q will fall on B if and only if AB=PQ. In this case PQ and AB cover each other completely and we say that AB is congruent to PQ.

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Q 1. Imagine the situation when a group of students are comparing angles They are not agreeing about whether the angles at A and B are congruent or not?

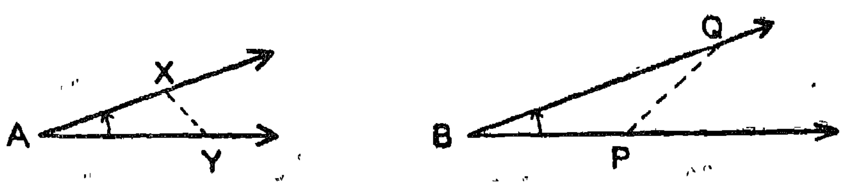



Fig. 5.5

(i) Anju says "$\angle$A is not congruent to $\angle$B. Since PQ > XY".

(ii) Manju says "It is wrong to compare PQ and XY since PQ is farther than XY from the vertex."

(iii) Ranjit says "the segments are equally far from vertex if AX is congruent to BP or AY is congruent to BQ".

How will you help these students to arrive at the right answer?

Q.2. You know that a triangle has six elements · three angles and three sides (segments).

If $\triangle$ PQR is congruent to $\triangle$ ABC, then :

(a) Will the three $\angle$s of $\triangle$ ABC be congruent to three $\angle$s of $\triangle$ PQR?

(b) Will the three sides (segments) of $\triangle$ ABC be congruent to three sides (segments) of $\triangle$ PQR?

Q.3. Is it possible to place a mirror between two segments of equal lengths so that the image of one segment will fall on the other?

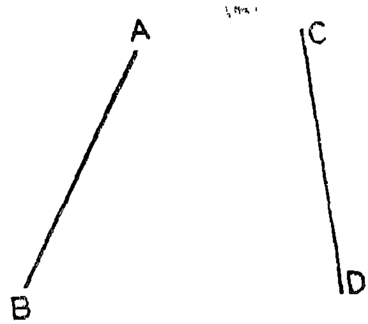



Fig. 5.6

If yes, how ? If no, why ?

---

(c) *Conditions for Congruence of Triangles*

We know that there are six elements in a triangle : The three sides and the three angles. Let us consider △ ABC and △ PQR.

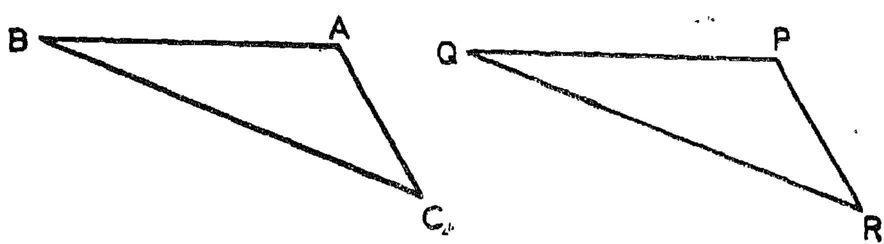



Fig. 5.7

Six elements of △ ABC are
∠A, ∠B, ∠C, AB, BC, CA.
Six elements of △ PQR are
∠P, ∠Q, ∠R, PQ, QR, RP.

If two triangles are congruent, then six elements of one triangle are congruent to the corresponding six elements of the other triangle.

Now the question arises—whether we need the congruence of all the six elements of one triangle and the corresponding six elements of the other triangle or congruence of fewer elements

We can ask our students to cut out a triangular shape from a piece of paper congruent to a given triangle and ask the students to observe the following :

(1) The sides (edges) of the paper triangle are congruent to the sides (edges) of the given triangle.
(2) The angles of the paper triangle are congruent to the angles of the given triangle.

(3) A pair of edges and the included angles of the paper triangle are congruent to those of the given triangle.
(4) A pair of the angles and the included side of the paper triangle are congruent to those of the given triangle.
(5) A pair of angles and a side of the paper triangle are congruent to those of the given triangle.

We know that to construct a triangle we need at least three elements. In the above five cases there are some cases in which we do not get a unique triangle. The students may be asked to construct triangles in each of the above five cases by taking particular values of the elements as given below :

(1) Construct a triangle whose three sides are AB$=$5 cm., BC$=$4 cm., CA$=$6 cm.
(2) Construct a triangle ABC whose three angles are $\angle A=30^\circ$, $\angle B=50^\circ$, and $\angle C=100^\circ$.
(3) Construct a triangle ABC so that side BC$=$5 cm., B$=30^\circ$, and C$=45^\circ$.
(4) Construct a triangle ABC so that AB$=$3 cm., BC$=$8 cm., and B$=80^\circ$
(5) Construct a triangle ABC, so that A$=30^\circ$, B$=80^\circ$ and side AC$=$6 cm.

Let a student observe the triangles constructed by his other friends. In which of the five cases he is not likely to get congruent triangles while comparing his triangle with the triangles of his friends?

Now the students will conclude the following conditions of congruence of triangles :

(1) Two triangles are congruent, if and only if the sides of one are respectively congruent to the sides of the other. This condition is symbolically denoted as SSS

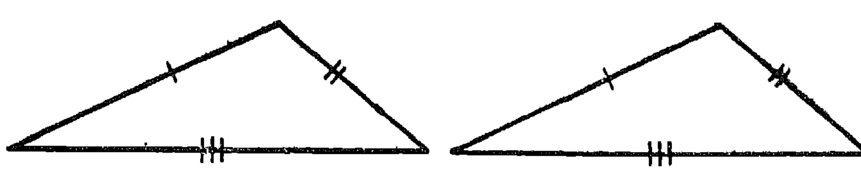


Fig. 5 8

(2) Two triangles are congruent if and only if a pair of sides and the included angle of one are respectively congruent to a pair of sides and the included angle of the other. This condition is symbolically written as SAS

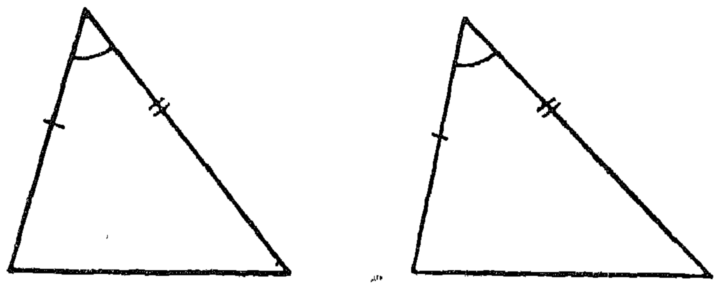


Fig. 5.9

(3) Two triangles are congruent if and only if a pair of angles and the included side of one are respectively congruent to a pair of angles and the included side of the other. This condition is symbolically written as ASA.

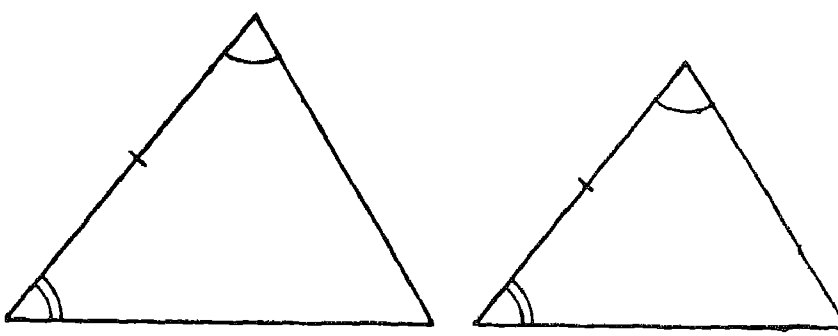


Fig. 5.10

(*d*) *Condition for Congruence of Two Right-angled Triangles*

If two triangles are right angled then all the above three conditions for congruence will hold good. Besides we have one more condition for their congruence.

"If the hypotenuse and a side of one right-angled triangle are congruent to the corresponding hypotenuse and the side of the other right-angled triangle, the two triangles are congruent". This condition is briefly written as RHS. This can also be verified by a student constructing a right-angled triangle with a given hypotenuse and a given side and then comparing his triangle with those of his class friends constructed with the same data.

---

Q. 1. Prove that two equilateral triangles, $\triangle$ ABC and $\triangle$ PQR are congruent if AB=PQ.

Q. 2. Name the congruent triangles in each of the following figures and state why they are congruent ?

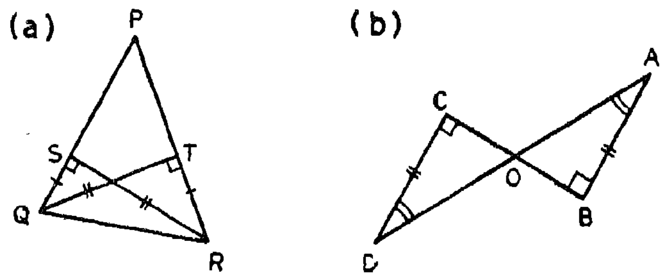



Fig. 5.11

### (3) Previous Knowledge Needed to Prove Congruence of Triangles

Whenever we have to prove that two given triangles are congruent, we have to prove any one of the following conditions:

(1) SSS
(2) SAS
(3) ASA
(4) RHS

To prove the above conditions we have to use previously learnt theorems. Some of them are :

(1) If two lines intersect, vertically opposite angles are equal.
(2) If two parallel lines are intersected by a transversal
  (i) alternate angles are equal
  (ii) corresponding angles are equal.
(3) The sum of three angles of a triangle is equal to 180°.
  Let us consider the following example :

1. If in a $\triangle$ ABC, AC=BC; $\angle$APM=$\angle$BPN and P is the mid-point of AB, prove that $\triangle$ APM $\cong$ $\triangle$BPN

**Proof**

We have to draw some inference from the given data.

$\because$ AC=BC

$\therefore$ $\angle$A=$\angle$B (Angles opposite to equal sides are equal)

In $\triangle$ APM and $\triangle$ BPN

$\angle$A = $\angle$B

$\angle$APM=$\angle$BPN [given]

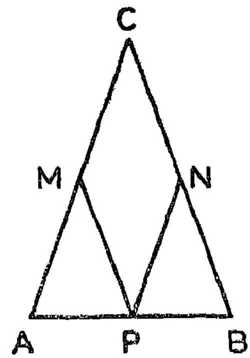



Fig. 5 12

AP=BP (∵ P is the mid-point of AB)

By ASA

$\triangle APM \cong \triangle BPN$

*Note* · Even to prove angles opposite to equal sides in a triangle are equal, we take help of the congruence of triangles.

### (4) Applications of Congruence of Triangles

Whenever we have to prove that (a) two segments are congruent, or (b) two angles are congruent, we have first to find out those triangles of which they are the elements and then prove that those triangles are congruent. If it is not possible to find out two such triangles, then by drawing suitable construction we can get two congruent triangles.

Look at the following examples :

**Example 1**

In the adjacent figure, AP and CQ are perpendiculars to the diagonal BD of the rectangle ABCD, prove that .

AP=CQ.

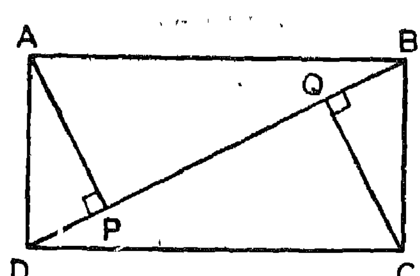



Fig. 5.13

Here you can see that $\triangle$ ADP and $\triangle$ CQB are two triangles of which AP and CQ are the elements respectively. To prove AP=CQ
We have to prove $\triangle$ ADP $\cong$ $\triangle$ CQB

*Note* : $\cong$ is the symbol of congruence.

**Example 2**

Prove that $\triangle$ ABC is isosceles if and only if altitude AD bisects BC. To prove $\triangle$ ABC is an isosceles triangle we have to prove AB=AC.

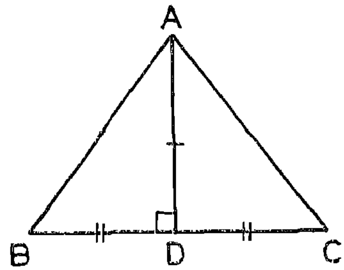



Fig. 5.14

To prove AB=AC, you can find two triangles $\triangle$ ABD and $\triangle$ ACD of which AB and AC are the elements respectively. To prove that AB=AC, we have to prove $\triangle$ ABD is congruent to $\triangle$ACD

**Example 3**

Prove that the chords of a circle, equidistant from the centre, are equal. ABDC is a circle such that OP=OQ.

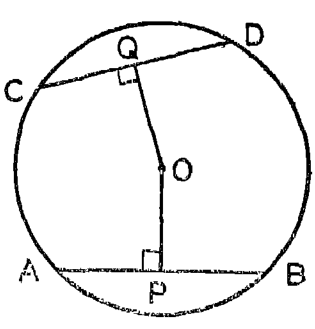



Fig. 5.15

We have to prove that AB=CD

To prove that AB = CD we have to find two triangles of which AB and CD are elements. We do not see any such triangle in the above figure. We have to do some construction to obtain two triangles of which AB and CD are elements.

This can be done by drawing radii from O to the points A,B,C,D, as below :

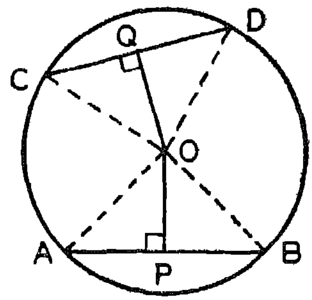



Fig. 5 16

Here we get two triangles $\triangle$ OAB and $\triangle$ OCD.

---

Q. 1. In the adjacent figure.
OP=OQ and OR=OS. To prove QR=PS, which congruent triangles will you choose?

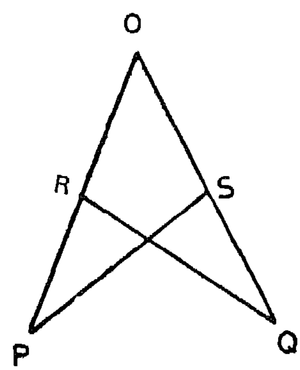



Fig. 5.17

Q. 2. To prove the statement "The perpendicular from the centre of a circle bisects the chord".
(a) Draw the figure.
(b) Name the triangles which you will take to prove that they are congruent.

Q. 3. "Equal chords subtend equal angles at the centre". To prove this draw a figure and write the names of triangles you will choose to prove the statement.

---

You have seen that congruence of triangles may be used as a tool to prove that (a) segments are equal and (b) angles are equal. Let us use this tool to prove the following statement :

If the bisector of a vertical angle of a triangle is at right angle to the base, the triangle is isosceles.

**Given**: ABC is a triangle. AD is the bisector of $\angle BAC$ which meets BC at D so that $\angle ADB = 90° = \angle ADC$. To prove : $\triangle ABC$ is isosceles, i.e., to prove $AB = AC$.

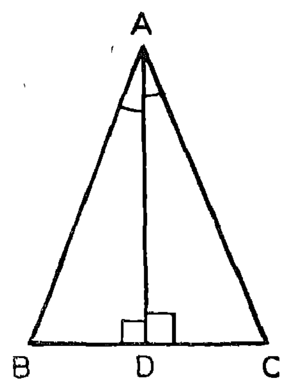



Fig. 5.18

**Proof**

$\triangle$ ABC is isosceles if $AB = AC$ i.e. if $\triangle ABD \cong \triangle CAD$ which is so because of ASA postulates as in $\triangle$ ABD and $\triangle$ CAD, $\angle BAD = \angle CAD$; $AD = AD$, $\angle ADB = \angle ADC$

Now you must have seen in the above example that we also use the properties of the figures which represent the hypothesis of the statements of the theorems to prove the congruence of triangles. As an additional illustration, consider the following example :

"Prove that the opposite sides of a parallelogram are equal".

**Given**: ABCD is a parallelogram, where $AB \parallel DC$ and $AD \parallel BC$. To prove $AD = BC$ and $AB = DC$.

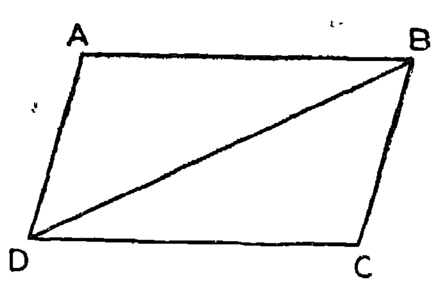



Fig. 5.19

As you already know to prove AD=BC, we need two congruent triangles of which AD and BC are elements. For this sake join BD.

**Proof** In $\triangle$ ADB and $\triangle$ BCD

$\angle$ ABD $=\angle$ CDB ($\because$ AB $\parallel$ DC and BD joins them. $\therefore$ Alternate angles).

Similarly $\angle$ ADB $=\angle$ CBD. Also DB $=$ DB By ASA. $\triangle$ ADB $\cong$ $\triangle$ BCD $\therefore$ AD $=$ BC. Similarly we can prove AB $=$ DC.

---

Q.1. Prove that ABC is isosceles if altitude AD bisects BC.

Q.2. ABCDEF is a six-sided figure having all its angles equal. Show that $\triangle$ ACE is an equilateral triangle.

---

**Teaching Strategies**

Congruence of triangles is used in proving many statements. In secondary classes, you should teach conditions of congruence as postulates. These can be related with the construction of triangles. Consider SSS as condition of congruence of triangles, then you can construct a unique triangle if three sides of a triangle are known.

More important is the use of congruence as a tool to prove many statements. In doing so you should emphasise the following points :

(1) To prove two segments or two angles to be equal, let the students associate those two segments or two angles with the appropriate congruent triangles.

(2) To prove that two triangles are congruent, let the students find out three suitable corresponding elements of each triangle.

(3) To prove two segments or two angles to be equal, let the students do some construction to find out two suitable congruent triangles.

(4) To prove two triangles to be congruent, let the students use some previously learnt theorem.

To impress upon the students the above points, you should ask the students to prove many statements. These statements should be arranged in order according to difficulty level. It will be useful if enough time is given to students to think for construction in proving the required statements.

**Evaluation**

Some of the objectives of this lesson are given below :

The students will be able :

(1) to state the conditions for congruence of triangles.

(2) to find out suitable elements in two triangles to prove their congruence,

(3) to devise the construction in a problem to obtain two suitable congruent triangles,

(4) to prove various statements involving the use of congruence of triangles

**Specimen Test Items**

(1) If AB and CD bisect each other at E, then prove that $\triangle ACE \cong \triangle BDE$.

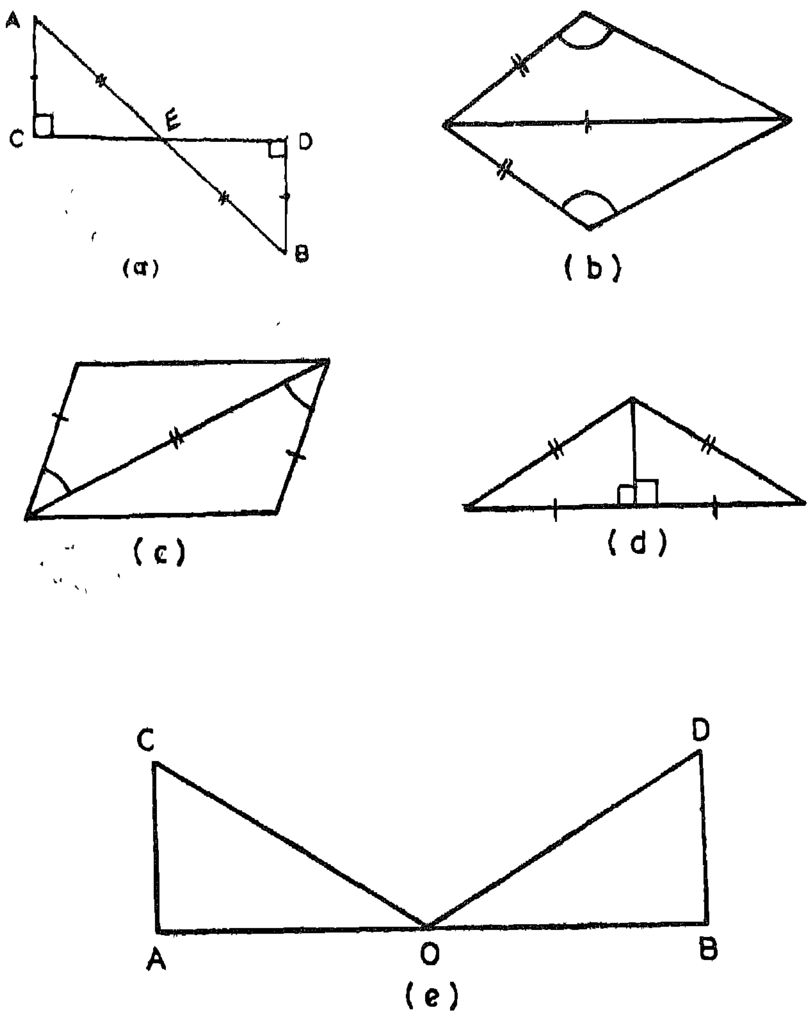



Fig. 5.20

Indicate which pairs of triangles can be proved congruent by using the SAS condition.

**Given** : AC $\perp$ OA; BD $\perp$ OB, AC = BD. [In Figure 5.20 (e)].
O is the middle point of AB
Prove that $\triangle$ OAC $\cong$ $\triangle$ OBD.

**Given**

ABC is an isosceles triangle such that AB = AC.
E and F are midpoints of sides AB and AC respectively. Prove that $\triangle$ CEB $\cong$ $\triangle$ CBF.

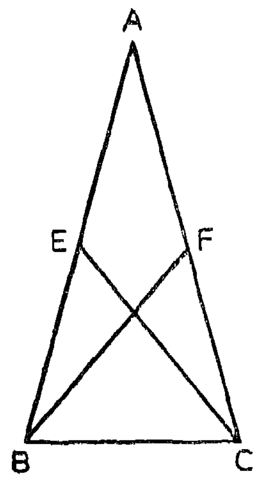



Fig 5..21

If AD bisects $\angle$ CAB; BD $\perp$ AB and DC $\perp$ AC prove that DB = DC.

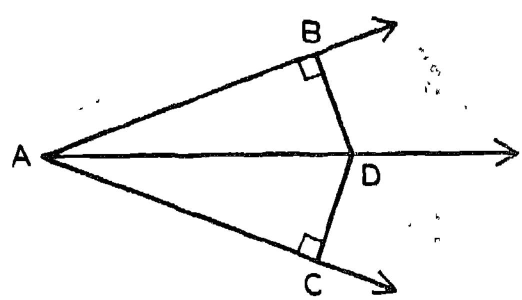



Fig. 5 22

**Assignment For Teachers**

(1) State five theorems where we use congruence of triangles to prove the statements.

(2) Give one example for each :

(i) A problem in which you have to prove that two triangles are congruent.

(ii) A problem where you have to prove two segments to be congruent by using congruence of triangles.

(iii) A problem where you have to prove two angles to be congruent by using congruence of triangles.

(3) State two problems where you have to do construction to obtain two congruent triangles or in the course of solving the problems.

(4) Do you use congruence of triangles in proving that the medians of a triangle intersect in the ratio 2 1 ? If yes, how will you help the students in devising construction?

(5) How does the knowledge of symmetry help in learning the congruence of triangles?

(6) Explain the method which you will use to find out the conditions for the congruence of two triangles.

## SIMILAR TRIANGLES

**Introduction**

We have seen some examples of similar figures in the lesson on congruence of triangles. We have categorized the figures of same shape and different sizes as a class of similar figures. In general it is not necessary that two figures must be of different sizes to be similar but they must be of the same shape.

For example: Any two circles are similar, any two squares are similar and any two equilateral triangles are similar.

---

Q. 1. Are any two circles congruent ?

Q. 2. Are two congruent figures always similar ?

---

The idea of "similar figures" is implicit when we talk of:

(1) A photograph and its enlarged size.

(2) A slide and its projected image.

From these examples, it is clear that there exists a correspondence between two similar figures in each case, precisely due to their magni-

fication properties. Now the question arises—given any two figures, how much of information we need to assess that they are similar We will try to find out the conditions for similarity between two triangles and their application in this lesson.

Q. Observe the following pairs and find out those pairs in which one figure is the enlarged copy of the other.

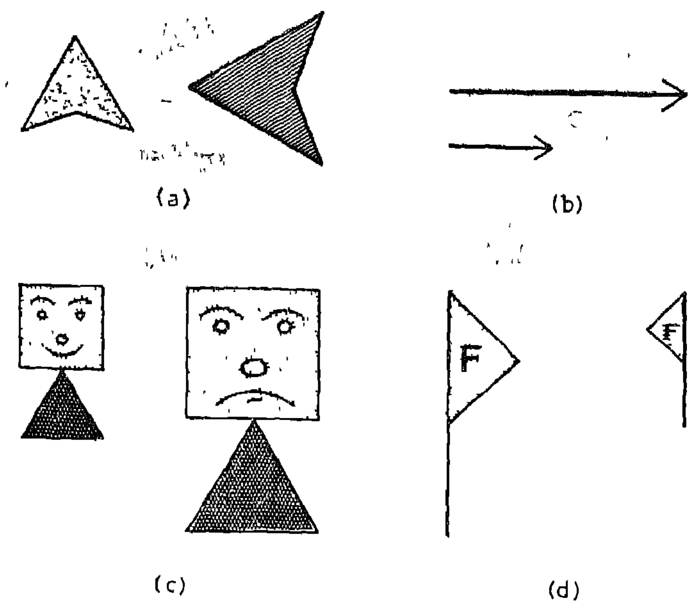



Fig. 5.23

**Content Covered in the Lesson**

(1) Concept of similar figures.
(2) Criteria for two triangles to be similar.
(3) Application of similar triangles in finding proportion of two segments.

**(1) Concept of Similar Figures**

Look at the following pairs of similar figures. Study the relations between their sides and angles ·

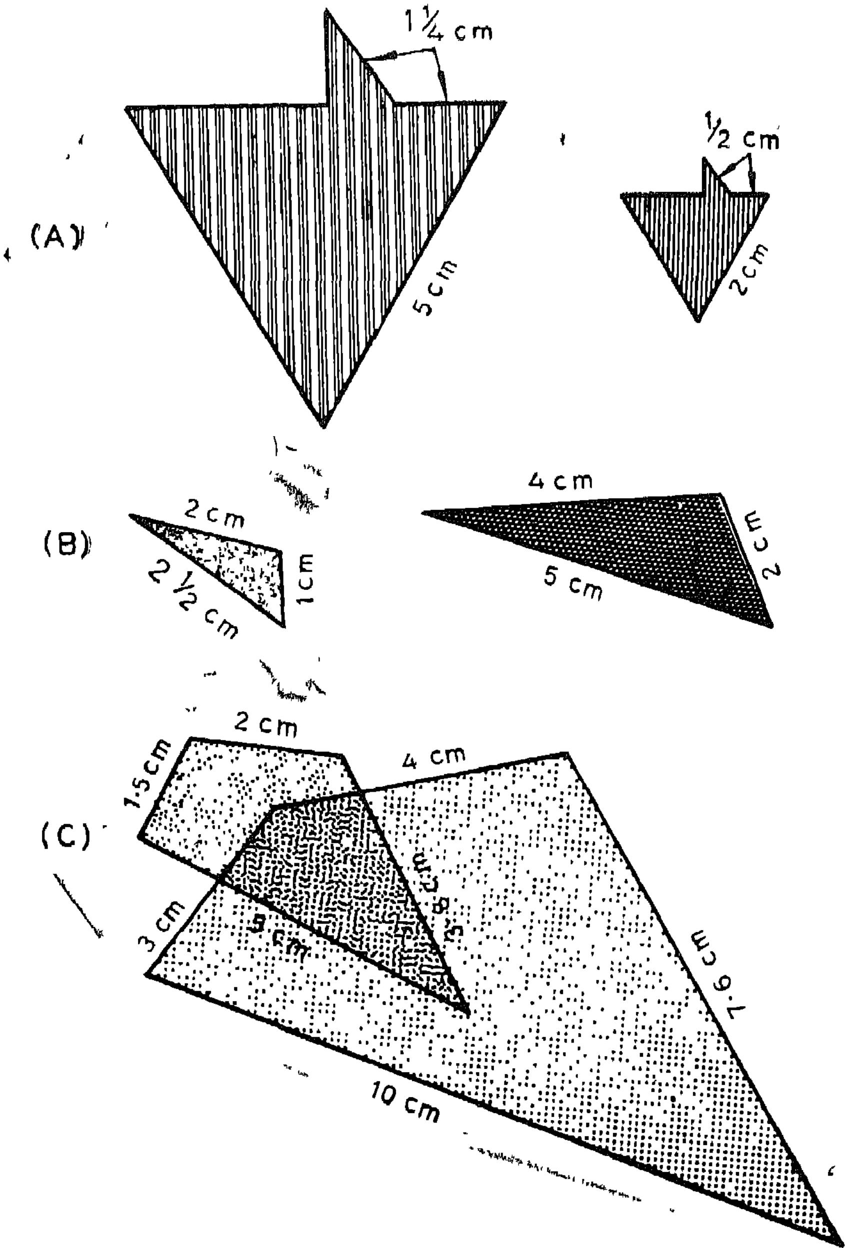

(A)
1¼ cm
5 cm
½ cm
2 cm
(B)
2 cm
2½ cm
1 cm
4 cm
5 cm
2 cm
(C)
2 cm
1.5 cm
4 cm
3.8 cm
5 cm
3 cm
7.6 cm
10 cm

Fig. 5.24

Look at the following pairs of figures and explain in what respect the pairs given below are different from the pairs of figures given on page 145

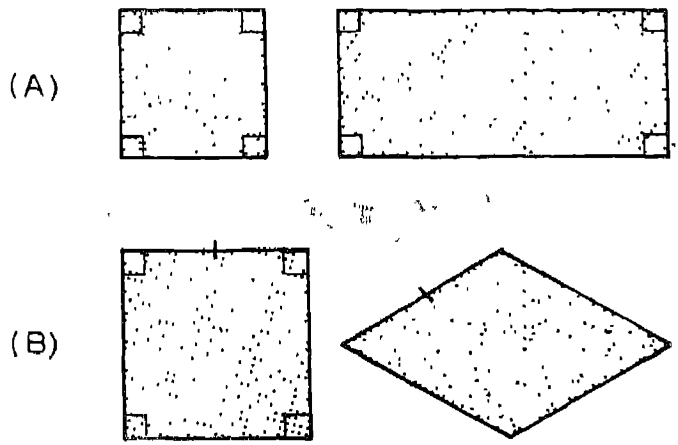



Fig. 5 25

You will observe that similar figures can overlap each other by magnifying or shrinking the figures. A rectangle cannot be shrunk into a square or a rohmbus cannot be shrunk into a square. Therefore they are not similar

You will also find the answer of the following questions by observing and comparing the above pairs of similar and non-similar figures.

(1) Is similarity affected by the position of the figures ?

(2) What can be said of the ratios of corresponding sides of two similar figures?

(3) Are similar triangles always equi-angular ?

(4) If two quadrilaterals are equi-angular, are they necessarily similar?

(5) If the corresponding sides of two quadrilaterals are in the same ratio, are they necessarily similar ?

You will conclude that for two figures to be similar (1) corresponding angles must be congruent and (2) corresponding sides must be proportional.

Thus if $\triangle$ ABC and $\triangle$ DEF are similar

then $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$. (Fig 5.26)

$$\text{and} \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

Just as in the case of congruence of triangles we may ask whether all these conditions are simultaneously needed for two triangles to be similar or only a few of them are needed, and others will get satisfied themselves.

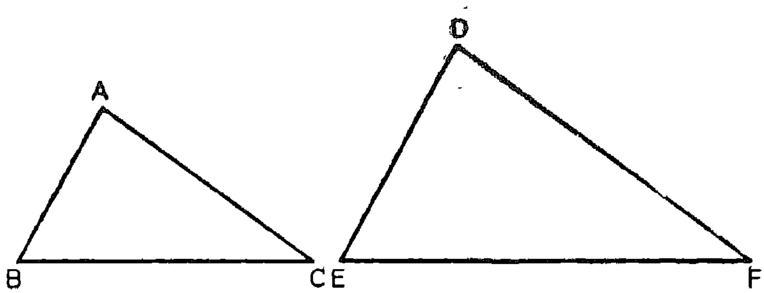



Fig. 5.26

**(2) Criteria for Similar Triangles**

By using the theorems on proportionality we can have the following three criteria for two triangles to be similar.

(1) The sides are proportional.
(2) Corresponding angles are equal.
(3) One angle of one is equal to one angle of the other, and the sides including these angles are proportional.

**(3) Application of Similar Triangles**

(a) You know that equiangular triangles are similar. This is a very important property. We use this for proving similarity between two triangles. Let us consider the following example :

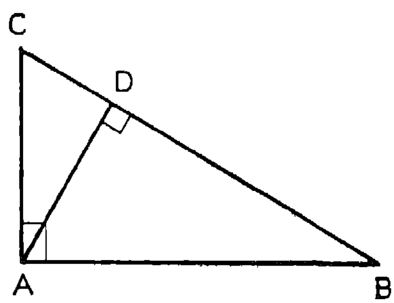



Fig. 5.27

ABC is a right-angled triangle. A is a right angle. Draw AD perpendicular on BC from A. To prove that ABC is similar to ABD. We will prove that ABC and ABD are equi-angular.

**Proof** : In $\triangle$ ABC and $\triangle$ ABD, we have

$\angle ABC = \angle ABD$

$\angle CAB = \angle ADB$ Right angle

$$\angle ACB = 90^\circ - \angle ABC$$
$$= \angle DAB$$

Hence $\triangle ABC$ and $\triangle ABD$ are similar

---

Q Prove that $\triangle$ ABC is similar to $\triangle$ ACD.

---

You have noted that to prove similarity of triangles, we have to prove that they are equiangular. Further, whenever the two ratios of segments are to be proved equal we take help of similar triangles. Look at the following example .

If a perpendicular AD is drawn from the vertex A of a right-angled triangle ABC to the hypotenuse BC, then BD : CD = AB : AC.

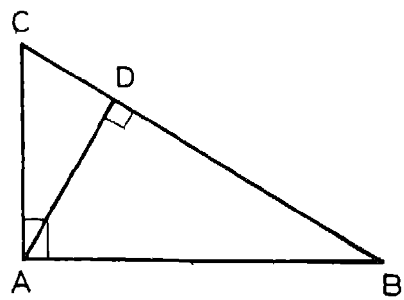



Fig. 5.28

We have proved earlier

$\triangle$ ABC is similar to $\triangle$ ABD and we know that ratio of corresponding sides in similar triangles is the same.

Therefore .

$$\frac{BC}{AC} = \frac{AB}{BD} \qquad \ldots \qquad (I)$$

Also $\triangle$ABC is similar to $\triangle$ ACD, therefore

$$\frac{BC}{AC} = \frac{AC}{CD} \qquad . \qquad (II)$$

From (I) and (II)

$$\frac{AB}{BD} = \frac{AC}{CD}$$

$$\frac{AB}{AC} = \frac{BD}{CD}$$

Hence BD : CD = AB : AC.

Q 1. Assuming that equiangular $\triangle$ s are similar, prove that if a line is drawn parallel to one side of a triangle, the other two sides are divided in the same ratio. Hint: To prove the statement, proceed according to the following steps

STEP 1. Construct figure.

STEP 2 Find two triangles and prove them to be equiangular.

STEP 3. Use the property of the ratio of sides of similar triangles.

Q. 2. Prove that EFD is similar to BCE in the following figure :

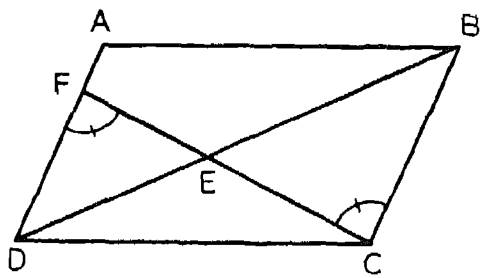



Fig. 5.29

Q. 3. In figure 5 30, PQ is parallel to BC. If the ratio AP : PB=2 3. Find the ratio QC . AQ.

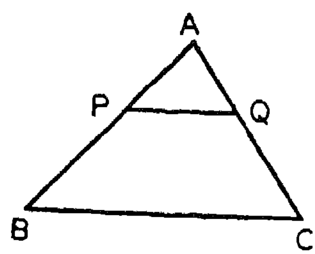



Fig. 5.30

You have seen that whenever we have to prove the equality of two ratios of segments, we use similar triangles. Sometimes to prove a statement we need similar triangles. In such cases, to obtain similar triangles, we have to do suitable construction.

Consider the following problem :

The bisector (internal or external) of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

You can see this problem is related to the ratio of sides and so to prove the statement we may need two similar triangles.

**Given :**

ABC is a triangle, AD is the bisector of BAC
Bisector AD meets BC at D.

**To Prove :** BD : CD = AB . AC

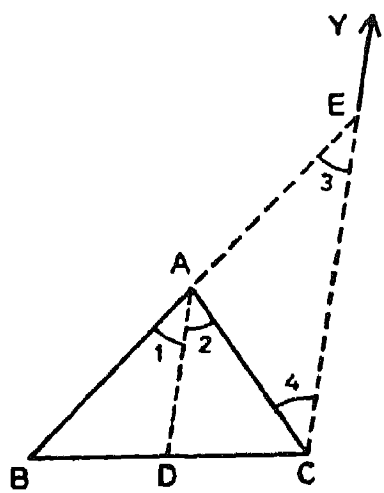



Fig. 5.31

*Construction :* To obtain similar triangles, draw CY parallel to DA. Extend BA to meet CY at E. We get two similar triangles · $\triangle$ ABD and $\triangle$ BCE.

**Proof** · Since AD $\parallel$ CE and BE is a transversal

$\therefore \angle 1 = \angle 3$ (corresponding angles)

Also AD $\parallel$ CE and AC is a transversal

$\therefore \angle 2 = \angle 4$ (Alternate angles)

In $\triangle$ ACE, AE = AC (sides opposite to equal angles)

From $\triangle$ ABD and $\triangle$ EBC

AB : AE = BD · DC

or AB : AC = BD : DC

Hence proved.

---

Q.1. Similarly prove the case of the above statement for the external bisectors of $\angle$ A.

Q.2. In the above problem, if a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides. Prove the above statement.

Q.3. State the converse of Q 2 and prove it.

**Evaluation**

Some of the objectives of teaching similar triangles are :

(1) The student can discriminate between similar figures and congruent figures.

(2) The student can prove that two triangles are similar if two angles of one are congruent to the corresponding angles of the other.

(3) The student can find the side of similar triangles if some of the sides of the triangle are known.

**Teaching Strategies**

By seeing the physical features of a girl and her mother, sometimes we say that she is a photocopy of her mother. Similarly we use the idea of similarity on many occasions in our life. This thing should be started in the beginning. After this we should use suitable geometrical diagrams to include the two conditions for similar figures .

(a) Angles are congruent.

(b) Corresponding sides of one figure to corresponding sides of other are equal.

To disprove that only one condition for similarity is not sufficient we can give the following counter examples.

**Example 1.** A square is not congruent to a rectangle.

**Example 2.** A square is not congruent to a rhombus.

You know that to disprove a statement only one example is sufficient and we call this the method of disproving a statement by a counter example

We should lay much emphasis on proving statements on the basis of similarity of triangles

In the beginning of congruence we should introduce similarity and congruence together. It helps the students to discriminate between similarity and congruence

**Specimen Test Items**

(1) Determine the pairs of corresponding sides of similar triangles; then find the length x in the following figure

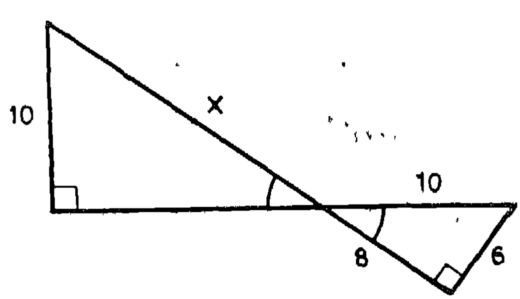



Fig 5.32

(2) Find x; if $\angle BAC = \angle DEC$

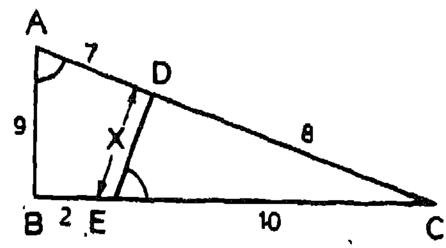



Fig. 5 33

(3) Prove that $\triangle$ ABD is similar to $\triangle$ ACE.

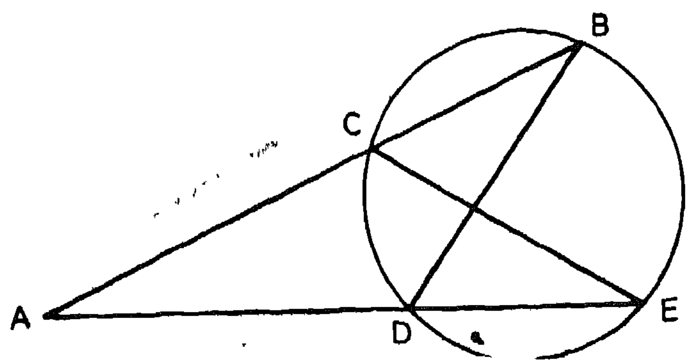



Fig. 5 34

(4) If two angles of $\triangle$ ABC are congruent to two angles of $\triangle$DEF, prove that they are similar.

(5) The areas of two similar triangles $\triangle$ABC and $\triangle$PQR are 60 and 84 sq. cm. respectively. If QR = 15.4 cm, find BC.

**Assignment for Teachers**

(1) A student was given a meter scale to measure the height of a building PQ. He has taken a photograph of the building along with the scale AB as shown in figure 5.35. Find out the height PQ of the building.
AB is meter scale.

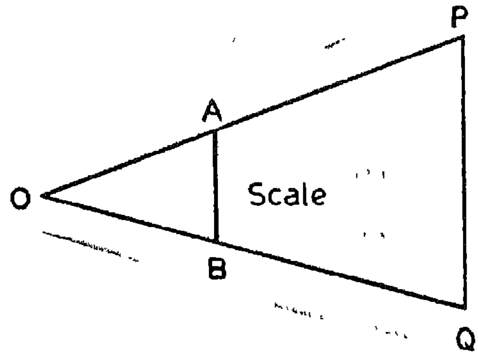



Fig. 5.35

(2) By joining the middle points of two sides in a triangle, prove that the new triangle formed will be similar to the original triangle.

(3) (a) Prove that all congruent triangles are similar.

(b) Prove that all similar triangles are not congruent.

(4) The angles in figure 5.36 marked with small squares are right angles.

Prove that (a) $\dfrac{BF}{BC} = \dfrac{AD}{AC}$

(b) Then show that :

$$\frac{BE}{AB} = \frac{CD}{AC} \times \frac{AC}{AB} + \frac{AD}{AC} \times \frac{BC}{AB}$$

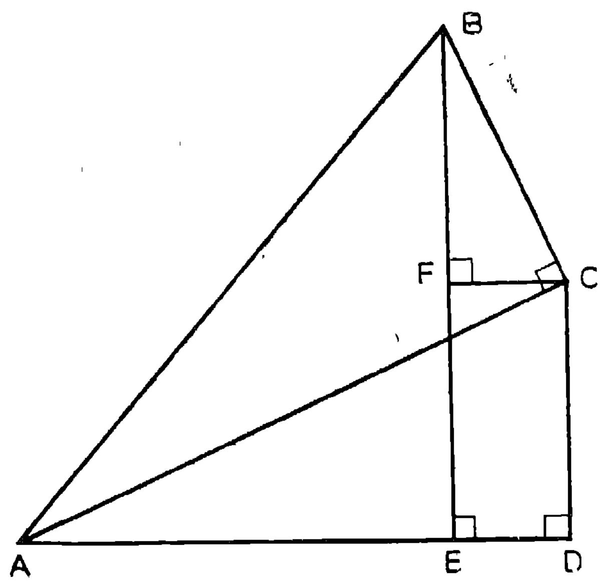



Fig. 5.36

(5) Prove that the ratio of the areas of two similar triangles is the square of the ratio of any two corresponding sides
(6) Mention some theorems so that you use the notion of similar triangles to prove them.
(7) Apply the similarity of triangles in finding a device to divide a segment into (i) two parts of the given ratio (say 2 3). (ii) three parts of the given proportion (say 2 3 4)

## CONSTRUCTION OF TRIANGLES AND CIRCLES

### Introduction

Perhaps it is a common view that practical geometry is easier to learn and understand than its theoretical counterpart. But it is not so. In fact, a full understanding of theoretical geometry is required for understanding practical geometry in its proper perspective. For example, the solution of a problem on construction of a triangle or a circle may require knowledge of the various related theorems. If one browses through a book on the history of mathematics, one will find that the motivation of the development of the geometrical theory stems from the practical need of geometry which often turns out to be problems relating to geometrical construction.

So in this lesson our approach will be to study, how the solution of a problem on construction of triangles and circles depends upon the knowledge of the properties of triangles and circles. Before starting this lesson the reader is advised to study the lesson on "The Nature and Scope of Mathematics"—particularly the part wherein we have discussed the problem solving technique by taking a problem on geometrical construction.

### Content Covered in This Lesson

(1) Simple problems in construction of triangles.
(2) Harder problems in construction of triangles.
(3) Construction of in-circle, circum-circle and ex-circle of a triangle.

### Development of the Concepts

*Construction of Triangles :* The problems on the construction of triangles can be divided into two types—(a) easy problems, and (b) harder problems.

In the case of easy problems of construction of triangles it will suffice if one tries to apply the following properties of triangles in the solution of the problem :

(a) Two sides of a triangle are together greater than the third. For example, a triangle with sides 3,5,10 is not a possibility.

(b) Postulates or theorems on congruence of two triangles (SSS, AAS, SAS), give clues regarding the uniqueness of a triangle and we conclude that a triangle can be usually drawn if (i) three sides or (ii) two angles and the included side or (iii) two sides and the included angle are given

In the case of a right triangle, hypotenuse and any one other side are sufficient data to construct the triangle

In the following cases no unique triangle can be drawn :

(a) Three angles are known (because three angles of a triangle are together equal to 180°) and so given two angles, the third can be known

(b) Two sides and the *acute* angle opposite to one of the two given sides are known. This is the ambiguous case in the solution of triangles.

The facts given above indicate, given certain data about a triangle, whether the triangle can be drawn As we have seen, the data may be insufficient for the construction of a unique triangle or inconsistent.

Let us now take a problem for illustration .

**Problem 1**

Construct a triangle ABC where $a = 5$ cm, $b = 4$ cm, $c = 4$ cm

Here the three sides of the triangle are given, so according to SSS postulate, a unique triangle is possible. But we have also to examine whether the sides are such that two sides are always greater than the third side. Now we observe that $4+4 > 5$, $5+4 > 4$, $5+4 > 4$. So the triangle can be constructed with the given data. Please note that here strict inequality is necessary In case of equality it is possible that the triangle degenerates into a straight line.

A useful hint for the construction for a right triangle, when two sides are given, is to apply Pythagoras theorem to find the third side and then to examine the possibility of the construction of the right triangle

Once a secondary student knows in what sequence a line (or an arc), or an angle of given magnitude are to be drawn, he is able to complete the actual construction But the ability to arrive at the knowledge of the above-mentioned sequence of construction depends upon one's knowledge of theoretical geometry.

---

Q Discuss the possibility and the method of drawing the following triangles.

1. ABC, where $a = 5.4$, $b = 4.5$, $c = 10$.
2. ABC, where $a = 5$, $b = 4$, $c = 3$.
3. ABC, where $a = 4.2$, $b = 3.7$, $C = 45°$.
4. ABC, where $a = 5.3$, $\angle B = 60°$, $\angle C = 100°$.
5. ABC, where $\angle A = 90°$, $a = 7.5$, $b = 4.3$

---

*Harder Problems in Construction of Triangles* We have so far discussed the easier problems in the construction of triangles. In the harder problems of construction of triangles, we have to adopt the technique of problem-solving which has four stages—

(a) Analysing or understanding the problem,
(b) Developing a plan for solution,
(c) Carrying out the plan, and
(d) Verification of the solution.

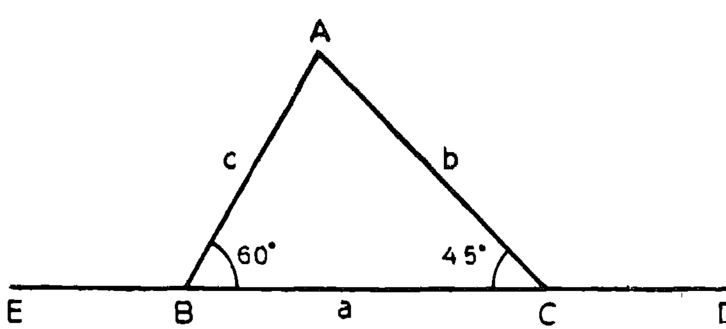
We will take some examples for illustration.

**Example 1**

Construct a $\triangle$ ABC such that $a+b+c=10$ cm
$\angle B=60°$, and $\angle C=45°$.

(a) **Understanding the Problem**

To start with we should draw a figure of a triangle for analysing the data.




ED = 10 cm.

Fig. 5.37

(b) **Devising a Plan for Solution**

To start with the problem does not lead to an easy solution. But we have to use the data $a+b+c=10$ cm. Some reflection on the problem will show that we have to start the construction with a segment of 10 cm and work in such a way that the side 'a' is cut out of this segment or is a part of this segment. Now we can carefully study the following figure ·

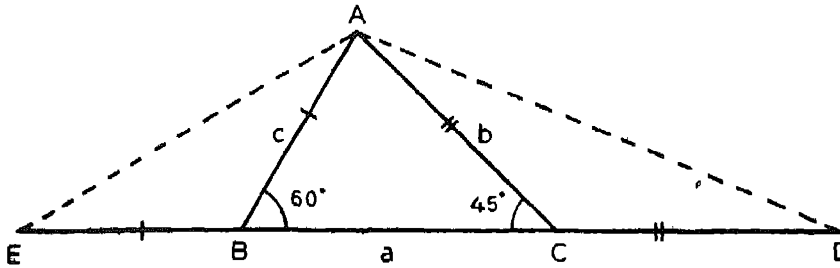



Fig. 5.38

Here we should have

EB=BA, DC=CA.

Consequently,

$\angle BEA = \angle BAE$, $\angle CDA = \angle CAD$

Also we note at this stage that

$$\angle BEA = \angle BAE = \tfrac{1}{2}\angle ABC = \frac{60^\circ}{2} = 30^\circ$$

$$\angle CDA = \angle CAD = \tfrac{1}{2}\angle ACD = \frac{45^\circ}{2}$$

This analysis gives us a method to construct the triangle ABC.

At the ends D and E of the segment DE we should construct angles of measures $\frac{1}{2}\angle ABC$ and $\frac{1}{2}\angle ACB$ so that the arms of these angles other than DE intersect at the point A. Then through A we draw AB and AC to meet DE in such a way that $\angle EAB = \angle BEA$ and $\angle DAC = \angle CDA$. Now $\triangle$ ABC is the required triangle.

(c) Carrying out the plan here means actually drawing the figure.

(d) Looking back at the problem we can surmise that a construction with the sort of data given in the problem should always be possible, because here the two arms of the angles at D and E (of measures $\frac{1}{2}\angle C$ and $\frac{1}{2}\angle B$ respectively), other than DE, are bound to meet at some point (say A).

**Example 2**

Construct a triangle ABC with $a=5.6$ cm., $b+c=10.2$ cm. and $\angle B - \angle C = 30^\circ$.

The following will be the stages of problem-solving in this case.

STAGE 1 · Understanding the Problem

Let us analyse the data given here The most important datum is $\angle B - \angle C = 30^\circ$ which gives us an insight into the nature of the shape of the $\triangle$ ABC as it indicates $\angle B > \angle C$. Now let us draw the following figure using the data and study it.

**Since $\angle B > \angle C$ $\therefore$ $b > c$. It is possible to draw the figure.**

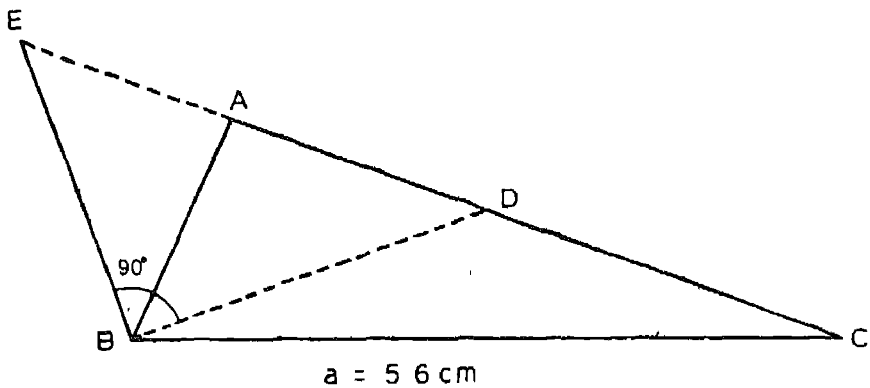



Fig 5 39

STAGE 2 · Developing a Plan for Solution

AD is cut out from AC for $b>c$. Here we have $AE=AB=BD$

So $\triangle$ EBD is a right triangle right angled at B so that $\angle EBD=90°$.

Also $EC=EA+AC=b+c$.

So if we are able to find the measure of $\angle EBC$ or $\angle DBC$, we can plan the device to construct the $\triangle$ ABC

Now since $AB = AD$, in $\triangle$ ABD,

$$\angle ABD = \angle ADB = 90° - \frac{A}{2}$$

So $\angle BDC = \angle ABD + \angle BAD$

$$= 90° - \frac{A}{2} + A = 90° + \frac{A}{2}$$

So in $\triangle$ DBC,

$$\angle DBC = 180° - \angle BDC - \angle DCB$$

$$= 180° - 90° - \frac{A}{2} - C$$

$$= \left(90° - \frac{A}{2}\right) - C$$

$$= \frac{B}{2} + \frac{C}{2} - C \text{ (in } \triangle \text{ ABC)}$$

$$= \frac{B}{2} - \frac{C}{2} = \frac{B-C}{2}$$

$$\text{So } EBC = 90° + \angle DBC = 90° + \left(\frac{B-C}{2}\right)$$

(In the calculations above A,B,C are taken as the measures of $\angle A$, $\angle B$, $\angle C$ of $\triangle$ ABC respectively.)

Once we find the measure of $\angle$EBC, we are in a position to devise a method for construction as follows :

(1) Draw BC = 5.6 cm

(2) At B draw BE in such a manner that

$$EBC = 90^\circ + \left(\frac{B-C}{2}\right).$$

(3) From C draw CE in such a manner that CE = b+c = 10.2 cm

(4) Draw BA in such a manner that $\angle$EBA = $\angle$BEA with A on CE. Now $\triangle$ ABC is the required triangle.

STAGE 3 . Carrying out the plan means actually drawing the construction.

STAGE 4 · Looking back at the solution we note that the data $\angle B - \angle C = 30^\circ$ are very important for the analysis of the problem. Also it is not always possible to construct the triangle ABC with given sort of data if b+c is not suitably given This is because in a triangle two sides of a triangle are together greater than the third side. For example, b+c = 5 cm is less than a = 5.6 cm and so with a = 5.6 cm b+c = 5 cm we cannot construct the $\triangle$ ABC.

---

Q. Discuss the stages of problem-solving technique in the solution of the following problems ·—

1. Construct a triangle ABC with a = 7.2 cm, b — c = 3.8 cm and $\angle B - \angle C = 45^\circ$.

2. Construct a triangle ABC where AB = 5 cm, AC = 4 cm and the median AD = 3 5 cm

3. Construct a triangle ABC where $\angle B = 45^\circ$, $\angle C = 60^\circ$ and AD = 2 cm, AD being perpendicular to BC from A.

4 Construct a triangle ABC with a = 5 cm, c = 3 cm, $\angle ABC = 120^\circ$, where E is the mid-point of AC.

---

**Construction of Circles Connected with a Triangle**

Following are the circles connected with a given triangle :

(1) Circum-circle

(2) In-circle

(3) Three ex-circles

Circum-circle of △ ABC

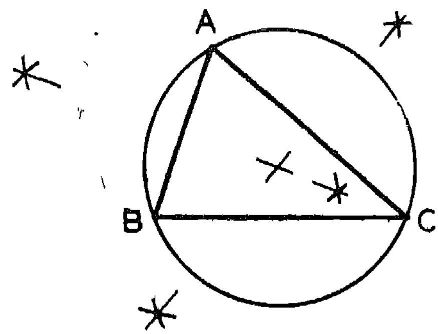



Fig. 5 40

In-circle of △ ABC

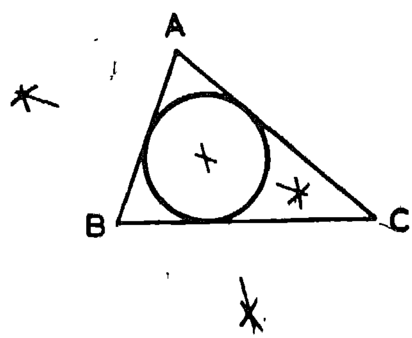



Fig. 5 41

The method of construction of these circles depends upon the following theorems .

(1) "Any point on the perpendicular bisector of the segment AB is equidistant from the points A and B." This theorem gives us a method to draw the circum-circle of a triangle ABC.

(2) Any point on the bisector (external or internal) of an angle BAC is equidistant from the straight lines AB and AC. This theorem gives us the method to draw the in-circles or ex-circles of △ ABC.

You should note that while the centre of circum-circle is equidistant from three points, the centres of the in-circle and the three ex-circles are equidistant from three straight lines. This is the reason why we have to take help of two different theorems for devising the methods of constructing (a) the circum-circle of a triangle, and (b) the in-circle and 3 ex-circles of a triangle.

Three ex-circles of $\triangle$ ABC

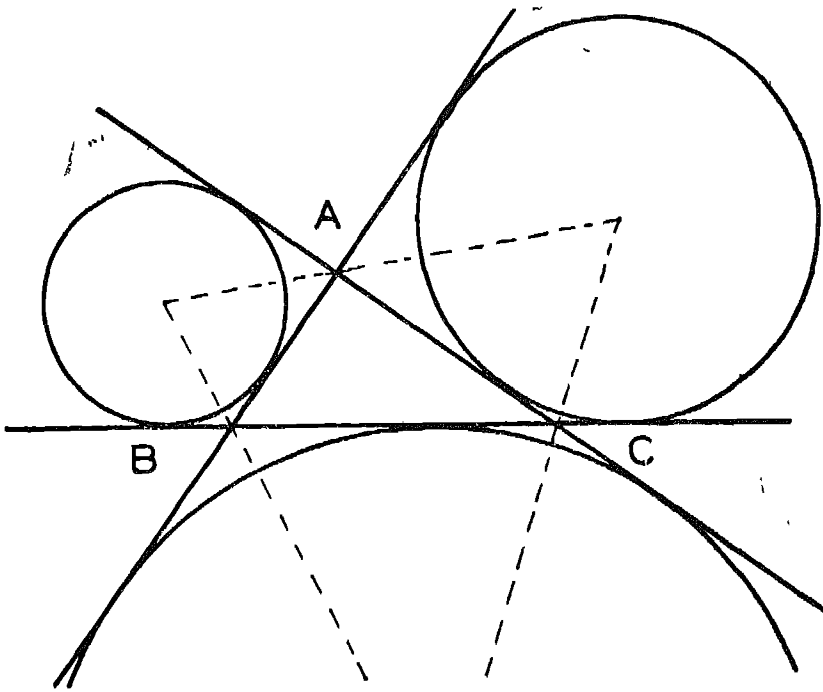



Fig. 5.42

It is interesting to discuss the problem of constructing circles touching the three sides of a triangle in a more general setting. Let us consider the following problem : "Analyse the construction of circles touching three given straight lines."

Here for an analysis of our problem, we should reflect on the relative positions of the lines Three cases emerge as given below :

(a) Three straight lines, $l_1$, $l_2$, $l_3$, may be parallel. In this case it is not possible to draw a circle touching the three straight lines

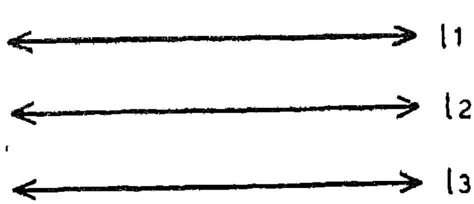



Fig. 5.43

(b) $l_1$, $l_2$ two of the three straight lines may be parallel, while the third one, i.e., $l_3$ intersects them. In this case only two circles can be drawn so as to touch the lines.

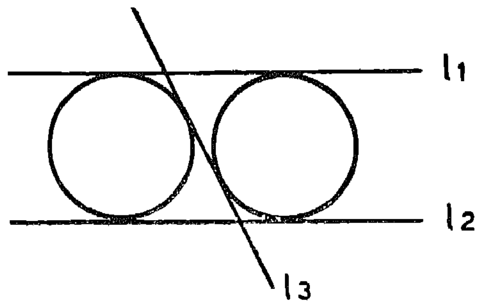



Fig 5 44

(c) Each pair of the three lines $l_1$, $l_2$, $l_3$, may intersect. In this case a triangle is formed by the lines $l_1$,$l_2$, $l_3$, and so the problem of construction is reduced to constructing the in circles and ex-circles of this triangle.

---

Q. 1. Analyse in detail, how the two theorems mentioned in this section help us in developing methods for constructing the circum-circle, the in-circle, and the three ex-circles of a $\triangle$ ABC.

Q. 2. Analyse the case of a quadrilateral for its circum-circle and the circle touching its sides.

---

**Teaching Strategies**

Generally teachers and geometry textbooks while dealing with the problems of construction give the methods of construction, but rarely say what is the rationale behind the methods, why the methods work and how we can arrive at the methods of construction. Though the skill of drawing the figures is an important objective of the study of geometry, yet more important is the skill in solving problems of construction which involves the four steps of problem-solving technique as described in Chapter I on the nature and scope of mathematics.

If this view of geometrical construction is taken, then any class of geometrical construction turns out to be a problem-solving session. This is the reason why we have emphasised, given a problem of construction and how to devise a plan for construction. This involves more mental skill rather than the skill in handling mathematical instruments. It should be noted that in this approach of teaching construction a thorough knowledge of theoretical geometry (related theorems and riders) is a prerequisite. For example, a knowledge of the theorems on congruence of triangles and a knowledge of the two theorems (concerning points equidistant from the two points of two straight lines) are required prerequisites for the

construction of triangles and circles respectively. Other theorems may also be needed for a more complete analysis of a problem and its solution.

It is not to say that students need not develop the skill to draw the figures by using mathematical instruments. In fact, this is very desirable and teachers should see that students develop this skill in appropriate measure. But the teacher should remember that the main objective of teaching geometry (which develops reasoning power in students) can be achieved only if in teaching geometry, the inter-relationship between practical and its theoretical counterpart is brought into focus whenever an occasion demands. Also, solving one problem on construction by a student himself is more effective than the teacher solving a number of problems. For this reason, while teaching construction the teacher should encourage the students to solve the problems by themselves so that the students get enough practice in this regard.

**Evaluation**

Some of the instructional objectives of this lesson are the following. The students will be able to :

(1) analyse a problem on construction of triangles and circles,
(2) devise a method for construction of a triangle or a circle with the given data.
(3) develop the skill of problem-solving with regard to problems of construction of triangles and circles.

**Specimen Test Items**

(1) Construct $\triangle$ ABC with BC $= 6.3$ cm and $\angle B = \angle C = 60°$
(2) Construct $\triangle$ ABC with $a = 5$ cm, $\angle B = 60°$ and $\angle A = 75°$
(3) Construct $\triangle$ ABC with $a+b+c = 13.5$ cm and $a.b:c = 2\ 3{\cdot}4$.
(4) Construct a right $\triangle$ ABC with BC $= 8$ cm and $b = 5$ cm
(5) Construct $\triangle$ ABC with $AB + AC = 8$ cm, $BC = 3$ cm and $\angle B = 45°$.
(6) Construct $\triangle$ ABC with $CA = 4$ cm, $\angle C = 30°$ and $BC—BA = 1.2$ cm.
(7) Construct $\triangle$ ABC with $BC = 7$ cm, $b—c = 2$ cm. and $\angle B—\angle C = 30°$
(8) Construct $\triangle$ ABC with base $BC = 5$ cm, altitude $AX = 4$ cm, and median $AD = 4\ 5$ cm.
(9) Construct a circle touching all the sides of a regular hexagon.
(10) Construct a circle circumscribing a regular hexagon.

**Assignments for Teachers**

(1) Examine the case of drawing a circle touching all the sides of .
(i) a rhombus (ii) a rectangle,

(2) Examine the case of drawing a circum-circle of a quadrilateral.
(3) Construct two problems of construction of triangles, where problem-solving becomes essential to solve the problem.
(4) What are the advantages of using problem-solving technique in practical geometry ? Support your answer.
(5) What is a nine-point circle ?

## PROBLEMS ON SURFACE AREA AND VOLUME OF REGULAR SOLIDS, RECTANGULAR PARALLELOPIPED, CYLINDER, SPHERE AND CONE

**Introduction**

If you go to a provision store, you will find the containers of oil, tea, powder, shampoo etc. These are made of iron sheet, plastic-sheet and cardboard. Manufacturers of these containers would like :

(a) to use the minimum sheet for a particular amount of contents. Say, a cylinderical vessel is needed for 4 kilograms of oil. What will be the minimum amount of sheet needed to construct the vessel?

and

(b) to use a container which will have a deceptive look of being big, but will actually be of less volume.

In all these problems we must use the formula for surface area and volume. In this lesson we will discuss the problems of surface area and volumes of the different types of containers.

**Content Covered in the Lesson**

1. Surface area and volume of a rectangular parallelopiped

(a) Properties of a rectangular parallelopiped.
(b) Formulae for volume and surface area of rectangular parallelopiped.
(c) Units of surface area and volume.
(d) Cube and cuboid.
(e) Convention.
(f) Problem types.

2. Surface area and volume of cylindrical surfaces

(a) Cylindrical surfaces.
(b) Convention
(c) Properties of a cylinder
(d) Formulae for volume and surface area.
(e) Problem types.

3. Surface area and volume of a cone
(a) Right circular cone.
(b) Properties of a cone.
(c) Formulae for the surface area and volume.
(d) Relation between volume of a cone and that of a cylinder.
(e) Problem types.

4. Surface area and volume of a sphere
(a) Properties of a sphere.
(b) Difference between a circular and spherical object.
(c) Formulae for volume and surface area.
(d) Problem types.

**1. Surface Area and Volume of a Rectangular Parallelopiped**

*1 (a) Properties of a Rectangular Parallelopiped*

We can take a rectangular parallelopiped container containing tea, match-sticks or any other content. Examine it. It will look like Fig. 5.45

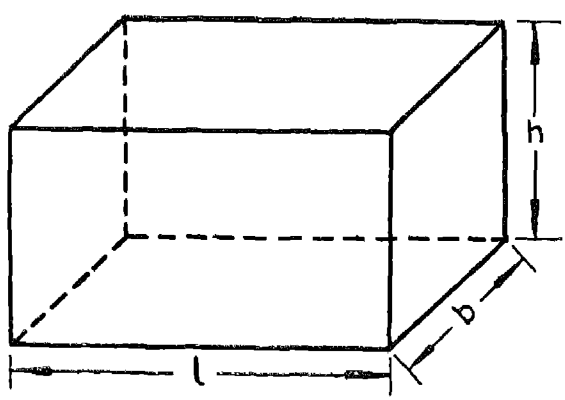



Fig 5 45

On observing we can infer:
(1) It consists of six faces.
(2) Each face is a rectangle.
(3) Opposite faces are congruent.

*1 (b) Formulae for Volume and Surface Area*

If l, b and h are tha length, width and height of a rectangular parallelopiped then surface area of six faces will be given by

$$S = 2(l \times b + h \times l + h \times b)$$

and volume V, will be given by

$$V = l \times b \times h$$

We can use the same formula for the volume to find out the volume of a rectangular pit, room, and a box But to find out the surface area of four walls or the surface area of an open box we have to change the formula for the surface. Area of four walls of a room :

$$= 2\,(l \times h + h \times b)$$
$$= 2 \times h\,(l + b)$$

This can be made clear by figure 5.46 or by actually seeing the four walls around you.

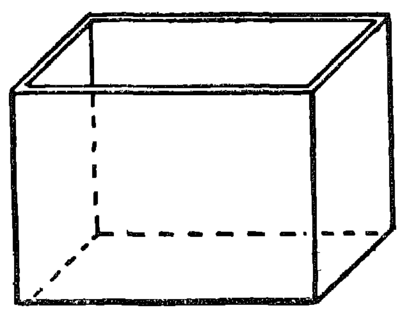


Fig 5 46

Area of the surface of open box ·

$$= 2 \times h \times (l + b) + (l \times b)$$

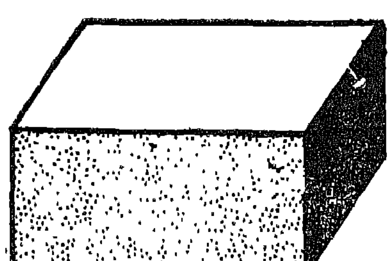


Fig. 5.47

Sometimes a box is made up of thick wood. In this case internal and external dimensions of the box are different. If "d" is the thickness of the wood and l, b and h are the length, width and height of the box, the internal dimension of a closed box will be l—2d, b—2d, h—2d. Look at the

following figure ·

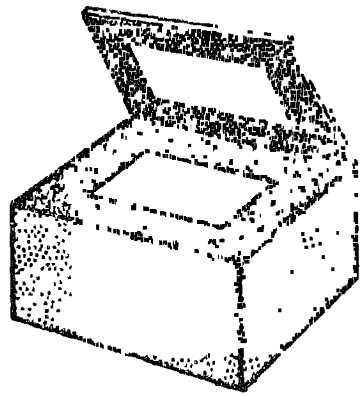


Fig 5.48

Volume of box$=(l-2d)(b-2d)(h-2d)$.

*1 (c) Units of Area and Volume*

If we measure l, b and h in cms , the unit of surface area will be square centimetre (sq. cms or $cm^2$) and unit of volume will be cubic centimeter (cc. or $cm^3$). While using the formula, it must be noted that l, b, and h must be in the same unit.

If they are not so, they must be changed into the same unit. The students often forget to convert l, b and h into the same unit.

We measure the volume of liquid in litres :

$$1 \text{ litre} = 1000 \text{ cc.}$$

$$1 \text{ cc} = \frac{1}{1000} \text{ litres.}$$

$$= 1 \text{ ml. (millilitre)}$$

"cusec" is a bigger unit of volume.

$$\begin{aligned} 1 \text{ cusec} &= 1\text{m} \times 1\text{m} \times 1\text{m} \\ &= 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm} \\ &= 1000000 \text{ cm}^3 \\ &= 1000 \times 1000 \text{ cm}^3 \\ &= 1000 \text{ litres.} \end{aligned}$$

Similarly "aar" is a bigger unit of area.

$$\begin{aligned} 1 \text{ aar} &= 10 \text{ m} \times 10 \text{ m.} \\ &= 100 \text{ sq. m} \end{aligned}$$

*1 (d) Cube and Cuboid*

Rectangular parallelopiped are called cuboids. If l, b and h of a cuboid are equal, i e , $l = b = h = a$ (say), the cuboid is called a cube.

Volume of the cube $= l \times b \times h$

$= a \times a \times a$

$= a^3$

Surface area of a cube $= 2\,(l \times b + b \times h + h \times l)$

$= 2\,(a \times a + a \times a + a \times a)$

$= 2\,(a^2 + a^2 + a^2)$

$= 6\,a^2.$

*I (e) Conventions*

A cuboid of length "l" width "b" and height "h" is represented dimensionally as $l \times b \times h$. For example a box of $1\frac{1}{2}$m $\times$ 1m $\times$ $\frac{1}{2}$m dimension is $1\frac{1}{2}$ m long; 1 m wide and $\frac{1}{2}$m in height

*I (f) Problem Types*

(1) *Direct Use of Formula* : If we know the length, width and height of a rectangular parallelopiped, we can find out its volume and area of its walls by directly substituting the values of l, b and h in the formula

*Note :* l, b and h must be in the same unit of length.

**Problem**

Find the capacity of a tank in litres, if its length, width and depth are 2 m, 3 m and $1\frac{1}{2}$ m respectively.

($1\ m^3 = 1000$ litres)

**Solution**

Capacity of the tank $= l \times b \times h$

$= 2 \times 3 \times \frac{3}{2}\ m^3$

$= 9\ m^3$

$= 9 \times 1000$ litres

$= 9{,}000$ litres

**Problem**

A classroom is 5 m $\times$ 4 m $\times$ 3 m. What is the area of its four walls ?.

**Solution**

Area of four walls $= 2 \times h\,(l + b)$

$= 2 \times 3\ (5+4)\ m^2$

$= 2 \times 3 \times 9m^2$

$= 54m^2$

---

Q. 1. Find the volume of a cube of side 10 cm.

Q. 2. How many cubic metres of earth can be removed in digging a pit 5 m long, 4 m wide and 2 m deep.

Q 3. How many cubic metres of concrete is needed to pour over a drive-way 9m long, 2m 40 cm wide and 10 cm deep?

Q. 4. What will be the area of cardboard needed to make a rectangular open box 12 cm long, 8 cm wide and 6 cm high ?

---

(2) *Direct Use of Formula and Associated Problems of Cost* · Labour charges are involved in painting a wall, digging a pit In such cases we must split the problem into two problems.

I. To find out area or volume.

II. To find out cost by using the result from the first.

**Problem**

If to dig a pit of size 1m $\times$ 1m $\times$ 1m the labour charges are Rs. 5 what would be the labour charges to dig a pit of 2m $\times$ 2m $\times$ 2m?

**Solution**

$$\text{Volume of the pit} = l \times b \times h$$
$$= 2m \times 2m \times 2m$$
$$= 8m^3$$

Labour charges of 1m $\times$ 1m $\times$ 1m is Rs 5

Labour charges of $1m^3$ is Rs. 5

Labour charges of $8m^3$ is Rs. $5 \times 8$

$= $ Rs. 40.00

---

Q. How many litres of paint will be needed to paint the walls and ceiling of a room 9m long, 6 m wide and 2.4 m high if 1 litre paint will cover 4 sq m?

---

(3) *Indirect Use of Formula* : We can derive many formulae from a given formula. For illustration, consider the following formula for volume of a cuboid :

$$V = l \times b \times h$$

$l \times b$ is the area of the base,
let it be denoted by A

$$V = A \times h \text{ ———————(1)}$$

$$\text{or} \quad h = \frac{V}{A} \text{ ———————(2)}$$

$$\text{or} \quad A = \frac{V}{h} \text{ ———————(3)}$$

We can solve many problems by using the above formulae (1), (2) and (3).

**Problem 1**

A rectangular rod is 10m long, its cross-section is a square of side 5cm. What will be its volume ?

**Solution**

$$\text{Volume} = A \times h$$
$$= \frac{5}{100} \times \frac{5}{100} \times 10 \text{ m}^3$$
$$= \frac{1}{40} \text{ m}^3$$

**Problem 2**

A cistern is constructed to hold 2 cusecs of water and the base of the cistern is a square meter. What is the depth of the cistern?

**Solution**

$$\text{Here } V = 2 \text{ cusecs}$$
$$= 2 \text{ m}^3$$
$$A = 1 \text{ sq. m.}$$
$$h = \frac{V}{A}$$
$$= \frac{2}{1} \frac{\text{m}^3}{\text{m}^2}$$
$$= 2 \text{ m.}$$

---

Q. 1 The volume of a tank is $1000\text{m}^3$ and its base is $50\text{m} \times 20\text{m}$. What will be its depth?

Q. 2. A rectangular wooden piece of dimensions 20 cm $\times$ 10 cm $\times$ 8 cm is cut out into 1 cm $\times$ 1 cm $\times$ 1 cm cubes. All cubes are joined together to form a rail. What is the length of the rail ?

---

(4) *Problem involving the use of the formula more than once :* We use carltons or crates to keep rectangular objects. For example, the crates are required to keep shoe-boxes. Question arises what should be the dimensions of a crate to keep 100 shoe-boxes of size 30 cm $\times$ 15 cm $\times$ 10 cm in it.

To construct such a crate we must keep the following points in mind.

(I). After putting 100 boxes, there should be no space left inside the crate, i.e , volume of the crate=volume of 100 shoe boxes.

$= 100 \times$ volume of one shoe box.

In general·

Volume of the crate=Volume of shoe box $\times$ no of boxes.

(II) Length, width and height of the crate must be multiples of 30 cm, 15 cm and 10 cm, i.e., multiples of the dimensions of the box.

(III) To prepare a crate we need wool planks, to use minimum amount of wool planks under conditions (I) and (II). We should try to adjust the length, width and height of the crate nearly of equal measure For example 30 cm $\times$ 15 cm $\times$ 10 cm $\times$ 100=Volume of the crate =(30 $\times$ 2) cm $\times$ (15 $\times$ 5) cm $\times$ (10 $\times$ 10) cm=Volume of the crate. Dimensions of the crate=60 cm $\times$ 75 cm $\times$ 100 cm.

Most of the problems are based on (I), where the number is to be calculated

$$\text{i.e., Number of boxes} = \frac{\text{Volume of the crate}}{\text{Volume of a shoe box}}$$

**Problem**

A brick measures 20 cm $\times$ 10 cm $\times$ $7\frac{1}{2}$ cm How many bricks will be required for a wall 25 m $\times$ 2 m $\times$ $\frac{3}{4}$ m.

**Solution**

$$\text{Volume of the wall} = 25 \times 2 \times \frac{3}{4}\ \text{m}^3$$

$$= \text{Volume of the brick} = \frac{20}{100} \times \frac{10}{100} \times \frac{15}{200}\ \text{m}^3$$

$$\text{Number of bricks required} = \frac{\text{Volume of the wall}}{\text{Volume of the brick}}$$

$$= \frac{25 \times 2 \times 3 \times 100 \times 100 \times 200}{4 \times 20 \times 10 \times 15}$$

$$= 25{,}000.$$

*Note:* It will be useful to convert into cm the dimensions of the wall instead of changing the dimensions of the brick into metres.

---

Q How many beams each 4 m $\times$ 20 cm $\times$ 12 cm can be cut from a piece of 12 m $\times$ 1 m $\times$ 80 cm?

---

(5) *Problems Where Construction is Needed*. There are some problems which you can easily comprehend by drawing figures

**Problem**

An open box is constructed by starting with a rectangular sheet of metal 20 cm × 28 cm and cutting a square of side 2 cm from each corner. The resulting projections are folded up and welded. What should be the volume of the resulting box?

**Solution**

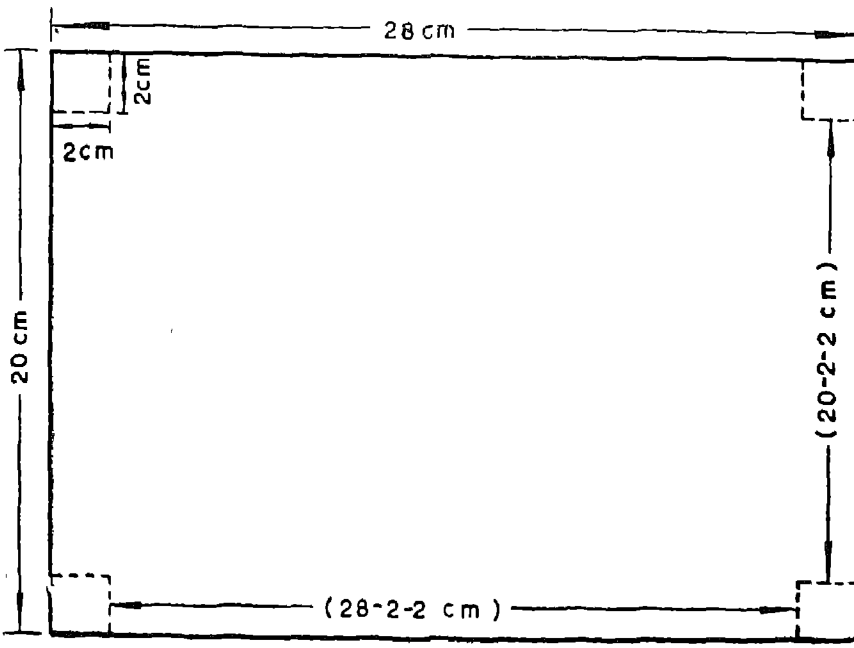



Fig. 5.49

drawing the above figure it can be easily comprehended that:

$l$ = Length of the box = (28—2—2) cm
= (28—4) cm
= 24 cm

$b$ = Width of the box = (20—2—2) cm
= (20—4) cm
= 16 cm

$h$ = Height of the box = 2 cm

Volume = $l \times b \times h$ = $(24 \times 16 \times 2)$ cm$^3$
= 768 cm$^3$

Q. Earth dug from a pit of 2 m × 1 5 m × 1m in a rectangular plot of ground 5 m × 3 m is spread over the rest of the same plot. What is the present depth of the pit?

(6) *Problems Involving both Area and Related Volume :* Read the following problem

**Problem**

What should be the volume of a rectangular solid whose side, front and bottom faces are 12 sq cm, 8 sq cm and 6 sq cm respectively?

**Solution**

You can think four steps of problem-solving to solve this problem·

STEP I: Find out what is given· Express it symbolically.

$$l \times h = 12$$
$$b \times h = 8$$
$$l \times b = 6$$

To find out volume.

STEP II: Devising a plan

We know that ·

$V = l \times b \times h$

We can find out $l^2 \times b^2 \times h^2$ by multiplying $l \times b$, $b \times h$, $h \times l$

STEP III· Carrying out the plan

$$V = l \times b \times h = \sqrt{l^2 \times b^2 \times h^2}$$
$$= \sqrt{l \times h \times b \times h \times l \times b} = \sqrt{12 \times 8 \times 6} = 24 \text{ cm}^3$$

STEP IV· Looking back

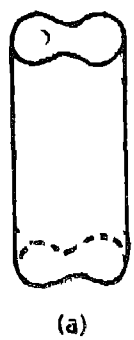
The volume of the box is equal to square root of the product of the areas of bottom, sides and front faces.

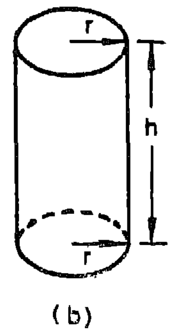
**(2) Surface Area and Volume of a Cylinder**

*2 (a) Cylindrical Surfaces*

You can see in figures 5.50, 5.51 and 5.52 that the bases of cylinders are congruent and have equal areas The bounding cylindrical surface is called the curved surface or lateral surface

The bases of cylinders (b) and (c) are circular, therefore we call them circular cylinders. A cylinder of type (b) in which the line joining the centre of bases is perpendicular to the base are called Right Cylinders.



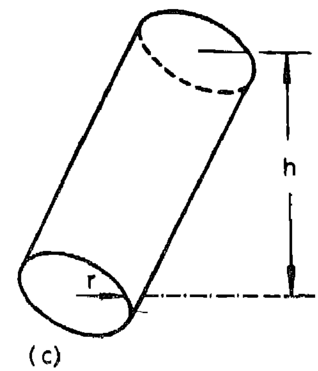




Fig 5.50 Fig 5.51 Fig 5.52

*2 (b) Convention*

When we say a cylinder, it means right circular cylinder.

*2 (c) Properties of a Cylinder:*

Cylindrical vessels are generally used to contain liquid objects We can take any cylindrical vessel and examine it. It will look like Fig. 5 53.

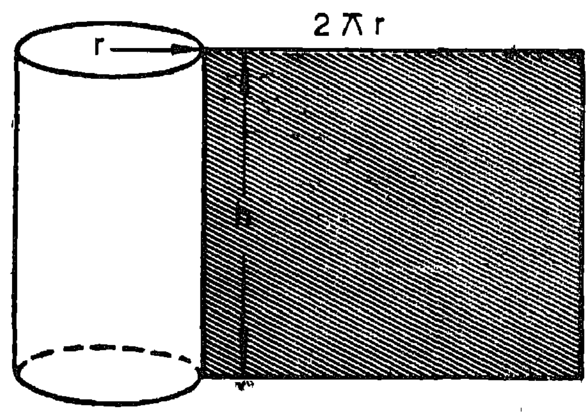



Fig. 5.53

On observing its properties, we can infer:

(1) It has two plane faces and one curved surface

(2) Two plane faces are circular and congruent.

(3) We can construct a cylinder by rolling a rectangular sheet

*2 (d) Formulae for Volume and Surface Area*

If r is the radius and h is the altitude of a cylinder, then volume:

$$V = \pi r^2 h$$

S = Curved surface of the cylinder

$= \text{Circumference of base} \times \text{altitude}$

$= 2 \pi r \times h$

$= 2 \pi r h$

and its total surface S will be given by

$S = \text{Curved surface area} + 2 \times \text{area of plane base}$

$= 2 \pi r h + 2 \pi r^2$

$= 2 \pi r (h + r)$

If the cylindrical box is hollow and open at one end, then the surface area will be given by:

$= 2 \pi rh + \pi r^2$

$= \pi r (2 h + r)$

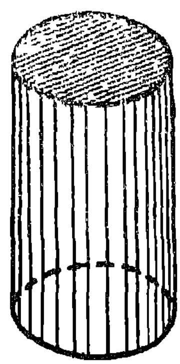


Fig 5.54

Look at the hollow cylinder. It is made of wood. What will be the

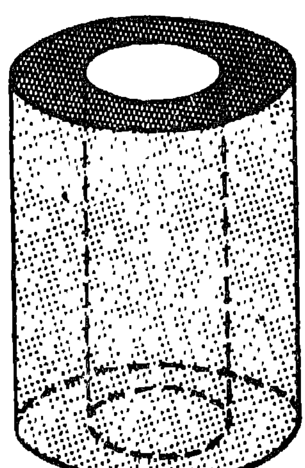


Fig 5 55

volume of the wood? Volume of the wood in hollow cylinder=Volume of external cylinder—Volume of internal cylinder.

$V = V_1 - V_2$

Where $V_1$ is the volume of external cylinder and $V_2$ is the volume of internal cylinder.

*2 (e) Problem Types*

(*1*) *Direct Use of Formula:* If the radius of the base and length are given, volume and area can be calculated by directly substituting the values of r and h in the desired formula. In some problems diameter may be given. We must reduce it to half to get radius.

---

Q. 1. What will be the volume of a cylinder of diameter 14 cm and length 100 cm?

Q. 2. Find the curved surface of a cylindrical box of radius 8 cm and height 28 cm.

Q. 3. Find the area of cardboard sheet needed to prepare an open cylindrical box of diameter 20 cm and height 20 cm.

Q. 4. Find the total area of the right cylinder with a radius 10 cm and height 7 cm.

---

(*2*) *Indirect Use of Formula:* We know that :

Volume of a cylinder = area of base × altitude

$$V = A \times h \qquad \text{————(1)}$$

$$h = \frac{V}{A} \qquad \text{————(2)}$$

$$\text{or } A = \frac{V}{h} \qquad \text{————(3)}$$

We can solve many problems by using the above formulae (1), (2) and (3). We have also to use previously learnt formula for area and circumference of a circle' surface area and volume of a rectangular parallelopiped.

**Problem 1**

How much will a km long wire of copper weigh if the area of its cross-section is .01 sq cm and density 8.5 gm/cc?

**Solution**

$$\begin{aligned}\text{Volume of the wire} &= \text{Area of base} \times \text{length}\\ &= .01 \times 1000 \times 100 \text{ cc}\\ &= 1000 \text{ cc.}\\ \text{Weight} &= V \times d \text{ gm wt}\\ &= 1000 \times 8.5 \text{ gm wt.}\\ &= 8500 \text{ gm wt.}\\ &= 8.5 \text{ kg wt.}\end{aligned}$$

**Problem 2**

A 100-cusec tank has a radius 2.1 m. What is its height?

**Solution**

$$\begin{aligned}\text{Height} &= \frac{\text{Volume of the tank}}{\text{Area of base}}\\ &= \frac{100 \text{ m}^3}{\frac{22}{7} \times 2.1 \times 2.1 \text{ m}^2}\\ &= \frac{700}{22 \times 4.41} \text{ m}\\ &= 7.21 \text{ m}\end{aligned}$$

---

Q. 1. A 1000 litre cylindrical vessel is 1 m high. What is its diameter?
STEP 1 Find the area of base 'A'
STEP 2: From $\pi r^2 = A$, find r.
STEP 3· Find 2 r to get diameter.

Q. 2. How many steel rods 10 cm in diameter and 14 cm in length can be made from a steel cuboid with dimension 140 cm $\times$ 160 cm $\times$ 200 cm?

Q 3. How many sq. m of asbestos sheet are needed to wrap 300 m of pipe of diameter 4 m.?

---

(3) *Problems Involving the Formulae of Both — Area and Volume* : Read the following problem

**Problem**

A rectangular sheet of paper of length 20 cm and width 11 cm is rolled along its width to form a cylinder. Find the volume of the cylinder so formed.

**Solution**

The length of the cylinder will be 20 cm and its width is equal to the circumference of the base or you can say that curved surface of the cylinder is equal to the area of the rectangle

$$2 \pi r h = \text{Area of rectangle}$$

$$\text{or } 2 \times \frac{22}{7} \times r \times 20 = 20 \times 11$$

$$\text{or} \quad r = \frac{20 \times 11 \times 7}{2 \times 22 \times 20} \text{ cm.}$$

$$= \frac{7}{4} \text{ cm.}$$

After finding out the radius of the cylinder, we can find its volume.

$$V = \pi r^2 h$$

$$= \frac{22}{7} \times \frac{7}{4} \times \frac{7}{4} \times 20$$

$$= \frac{385}{2}$$

$$= 192.5 \text{ c.c.}$$

What will be the number of circular pipes with an inside diameter of 1 cm which will carry the same amount of water as a pipe with an inside diameter of 6 cm?

(*4*) *Problems where Construction is Needed*· There are some problems which can be easily comprehended by drawing the figures. Read the following problem :

**Problem**

An iron pipe has an inside diameter 10 cm and is 1 cm thick. How much will 100 cm of the pipe weigh ? (Density of iron = 7.8 gm/cc),

**Solution**

The figure of its cross-section is :

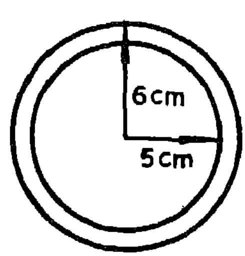



**Fig. 5.56**

Inside radius $= 5$ cm.

Outside radius $= 5$ cm $+ 1$ cm

$= 6$ cm.

$$V = V_1 - V_2$$

$$= \pi \times 6^2 \times 100 - \pi \times 5^2 \times 100$$

$$= \frac{22}{7} \times 100 \times (6^2 - 5^2)$$

$$= \frac{22}{7} \times 100 \times 11 \text{ c.c.}$$

Wt. of iron pipes $= V \times d$ gm wt

$$= \frac{22}{7} \times 100 \times 11 \times 7.84 \text{ gm wt.}$$

$$= \frac{22 \times 100 \times 11 \times 7.84}{7 \times 1000} \text{ kg wt.}$$

$$= 27.104 \text{ kg wt.}$$

---

Q. 1. Two circular cylinders of equal radius (3cm) are bound tightly together by an elastic so that the axes of the cylinders are parallel and the band lies in a plane perpendicular to these axes. Calculate the stretched length of the band.

Q. 2. A cylinder of maximum size is cut out from a cube of side 4 cm. What is the volume of the remaining portion ?

Q 3. A cylinder is divided into two equal halves by a plane perpendicular to the base. Find the total surface area of one of the halves if radius is 8 cm and height 10 cm.

---

## 3 Surface Area and Volume of a Cone

### 3 (a) *Right Circular Cone*

(I) A cone is said to be right circular cone, if the line joining the centre of the base to the vertex is at right angle to the base.

(II) Conventionally

We call a right circular cone, a cone only.

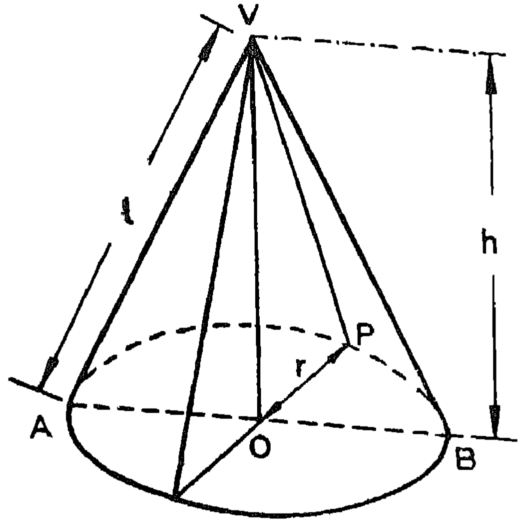



Fig. 5.57

*3. (b) Properties of a Cone*

(I) A cone has one plane (circular) surface and one curved surface.

(II) Vertical height h, radius r and slant height l are related as follows:

$$l^2 = h^2 + r^2$$

*3. (c) Formulae for the Surface Area and Volume of a Cone*

Volume of a cone $V = \frac{1}{3} \pi r^2 h$

Curved surface area of a cone $= \pi . r . l$

Total surface area of a cone $= \pi . r\ l + \pi r^2$

$= \pi . r . (l + r)$

*3. (d) Relation Between Volume of a Cone and that of a Cylinder*

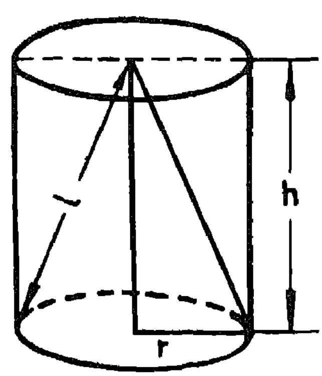



Fig. 5.58

We know that volume of a cylinder $= \pi r^2 h$

Volume of a cone of the same height 'h' and same radius 'r'

$$= \frac{1}{3} \pi r^2 h$$

Thus the volume of the cone is $\frac{1}{3}$ (the volume of the corresponding cylinder). This can be demonstrated.

*3. (e) Problem types*

*(1) Direct use of formula*

**Problem 1**

Find the volume of a right circular cone with slant height 5 m and an altitude 3m.

**Solution**

$$\text{Volume} = \frac{1}{3} \times \pi \times r^2 \times h$$

We need r and h, h is known and it is equal to 3 m, while r can be calculated by using the formula

$$l^2 = r^2 + h^2$$
$$5^2 = r^2 + 3^2$$
$$r^2 = 5^2 - 3^2 = 16$$
$$r = 4 \text{ m}$$
$$\text{Volume} = \frac{1}{3} \times \frac{22}{7} \times 4 \times 4 \times 3 \text{ m}^3$$
$$= \frac{352}{7} \text{m}^3$$
$$= 50.28 \text{ m}^3$$

---

Q. 1. Find the total area of the right circular cone having an altitude 12 cm and radius 5 cm.

Q. 2. Find the total area of the right circular cone having slant height of 30 cm and a base radius of 18 cm.

Q. 3. The circumference of the base of a cone is 66 cm and its slant height is 32 cm. Find the height of the cone.

---

*(2) Problems Based on Figures :* Sometimes it is difficult to state the problem in words only. In such cases we draw figures of the solid and mark the data in it.

**Problem**

V -- ABCD is a right circular cone as shown below .

(a) Find the lateral area of the frustum EACG.

(b) Find the total area of the frustum EACG.

(c) Find the volume of the frustum EACG.

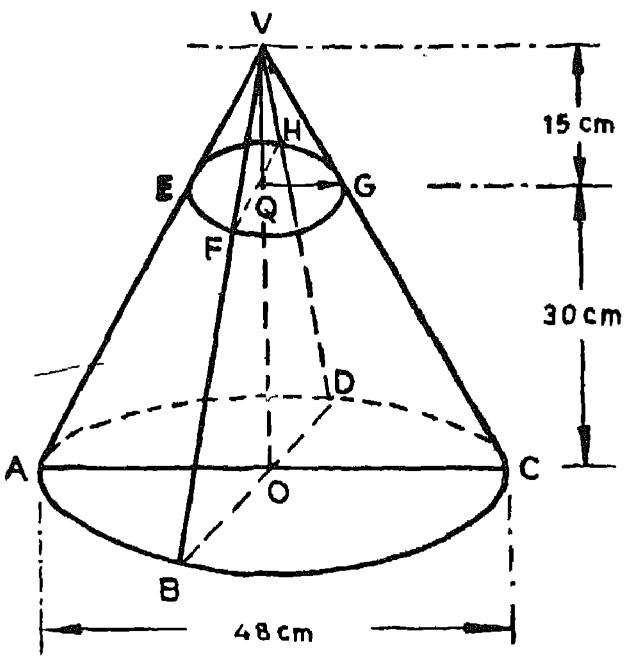



Fig. 5.59

**Solution**

To find out lateral surface we need slant height of the cones.

Slant height l of the cone V-ABCD

$$= \sqrt{h^2 + r^2}$$
$$= \sqrt{45^2 + 24^2}$$
$$= \sqrt{2025 + 576} = \sqrt{2601}$$
$$= 51 \text{ cm.}$$

By similar triangles V Q G and VOC

$$\frac{15}{45} = \frac{l_1}{51} = \frac{rl_1}{24} \text{ (Where VG} = l_1\text{, and QG} = r_1\text{)}$$

$$l_1 = \frac{15 \times 51}{45} = 17 \text{ cm.}$$

$$r_1 = 8 \text{ cm.}$$

Lateral surface area of the frustum = lateral surface area of the cone VABCD

— Lateral surface area of the cone V—EFGH

$= \pi \times 24 \times 51 - \pi \times 17 \times 8$

$= \frac{22}{7} \times 8 \times 17\,(9-1)$

$= \frac{22}{7} \times 8 \times 17 \times 8 \text{ cm}^2$

$= 3419.4 \text{ cm}^2$

(b) Total surface area of frustum :

= Surface area of slant surface + Surface area of base circles

$= 3419.4 \text{ cm}^2 + \frac{22}{7} \times 8^2 + \frac{22}{7} \times 24^2$

$= 3419.4 \text{ cm}^2 + 2011.4 \text{ cm}^2$

$= 5430.8 \text{ cm}^2$

---

Q. Find the volume of the solid PRSTLK.

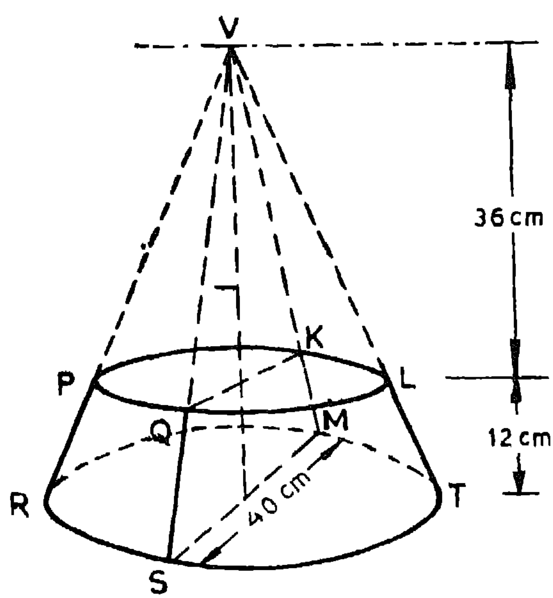



Fig. 5.60

(c) Volume of the frustum = Volume of cone V-ABCD

$$- \text{Volume of the cone V. EFGH}$$

$$= \tfrac{1}{3}\pi(24)^2 \times 45 - \tfrac{1}{3}\pi \times (8)^2 \times 15$$

$$= \frac{1}{3}\pi.8^2.15\,(27-1)$$

$$= \frac{1}{3} \times \frac{22}{7} \times 64 \times 15 \times 26$$

$$= 26148.5 \text{ cm}^3$$

*(3) Problems Involving the Volume and Surface Area of a Cylinder and a Cone:* Students usually make mistakes in choosing the correct formula. To correct this mistake we must include problems in which formula for volume and surface area of a cylinder and a cone are involved together.

**Problem 1**

The volume of a cone is equal to that of a cylinder 12 cm high and 30 cm in diameter. If the height of the cone is 108 cm, find its radius.

**Solution**

Let the radius of the cone be $= r$ cm.
Volume of the cone $= \frac{1}{3} \times \pi \times r^2 \times 108$
Volume of the cylinder $= \pi \times 15 \times 15 \times 12$
According to given conditions·

$$\tfrac{1}{3} \times \pi \times r^2 \times 108 = \pi \times 15 \times 15 \times 12$$

$$r^2 = \frac{\pi \times 15 \times 15 \times 12 \times 3}{108 \times \pi}$$

$$= 75$$

$$r = 5\sqrt{3} \text{ cm}$$

**Problem 2**

Find the canvas needed to stitch a tent as shown in Fig. 5.61.

**Solution**

Canvas will be used:
(a) to stitch a conical part.
(b) to stitch a cylindrical part.
(a) Area of the canvas $A_1$ will be equal to the area of the slant surface of the cone.

$$A_1 = \pi.\, r.\, l$$

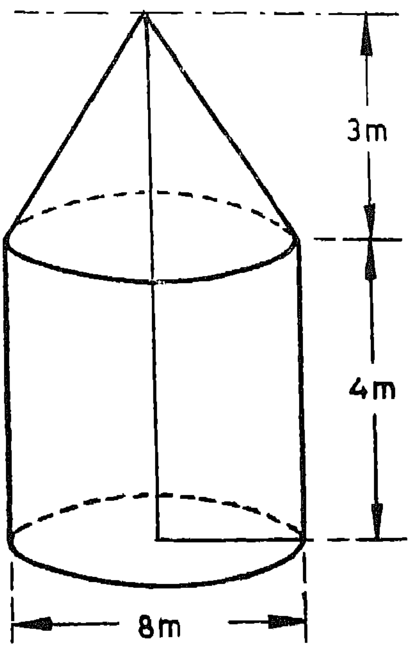



Fig. 5.61

here r $= 4$ m and

$$l = \sqrt{n^2 + r^2}\ m = \sqrt{16 + 9}\ m = \sqrt{25}\ m = 5m$$

$$A_1 = \frac{22}{7} \times 4 \times 5\ m^2$$

$$= \frac{440}{7}\ m^2$$

(b) Area of canvas $A_2$ will be equal to the area of lateral surface of the cylinder

$$A_2 = 2\ \pi\ rh$$

$$= 2 \times \frac{22}{7} \times 4 \times 1$$

$$= \frac{176}{7} m^2$$

Canvas needed $= A_1 + A_2$

$$= \left(\frac{440}{7} + \frac{176}{7}\right) m^2$$

$$= \frac{616}{7} m^2$$

$$= 88\ m^2$$

Q. 1. Find the volume and surface area of the following solid

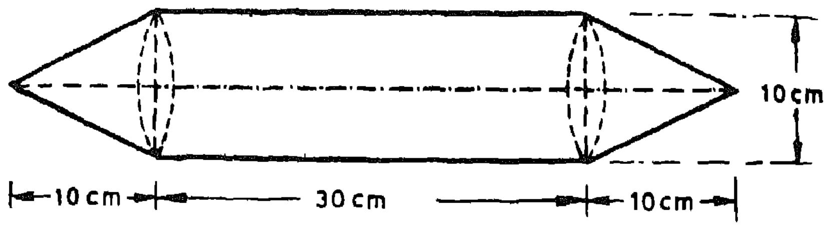



Fig. 5.62

Q. 2. A solid right circular cylinder of radius 7 cm and height 10 cm is melted into a right circular cone of radius 7 cm. Find its height. (Do not make any computation).

Q. 3. Find the length of the canvas 2 m in width, required to make a conical tent 8 m in radius and 4 m in slant height.

## (4) The Volume and Surface Area of a Sphere

### (a) *Properties of a Sphere*

Look at a ball. It is spherical.

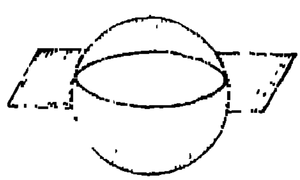


Fig. 5.63

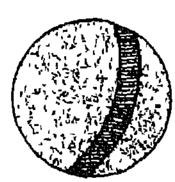


Fig. 5.64

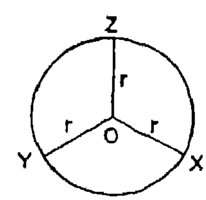



Fig 5.65

A sphere has only one curved surface. The distance of any point of the sphere from its centre is the same. A plane intersects a sphere in a circle.

### (b) *Difference between Circular and Spherical Objects*

A disc is circular while a ball is spherical. A circular object is two dimensional while a spherical object is three dimensional. But still on a paper we represent a spherical object by drawing a circle; when a circular figure represents a spherical object, will be clear from the context.

### (c) *Formulae for Volume and Surface Area of a Sphere*

Surface area of a sphere $= 4\pi r^2$

Volume of a sphere $= \frac{4}{3}\pi r^3$

If a sphere is divided into two equal halves as below:

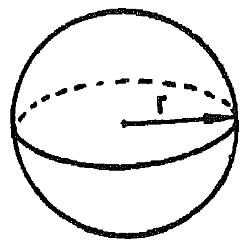



Fig. 5.66

A sphere is divided by a plane through its centre into two hemispheres. A hemisphere has two surfaces—a plane circular surface and a curved surface.

Total surface of a hemisphere

$$= 2\pi r^2 + \pi r^2$$
$$= 3\pi r^2$$

*(d) Problem Types*

*(1) Direct and Indirect Use of Formula.* You can calculate the volume and surface area of a sphere if the radius is known, by using formula.

$$V = \frac{4}{3}\pi r^3$$
$$S = 4\pi r^2$$

From these formulae, we have

$$r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \qquad \text{------------(1)}$$

$$r = \left(\frac{S}{4\pi}\right)^{\frac{1}{2}} \qquad \text{------------(2)}$$

By using the formulae (1) and (2), we can find the radius.

**Problem**

What is the radius of a sphere of volume 84.82 cc?

**Solution**

$$\text{Here}\quad V = 84.82 \text{ cc}$$

$$= \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$$

$$r = \left(\frac{3 \times 84.82}{4 \times 3.14}\right)^{\frac{1}{3}}$$

On taking logarithm

$$\log r = (\log 3 + \log 84.82 - \log 4 - \log 3.14)$$
$$= (.47712 + 1.92848 - .60206 - .49693)$$
$$= (1.90661)$$
$$= .63554$$

on taking antilog

$$r = 4.322 \text{ cm.}$$

---

Q. 1. Find the volume of a sphere of diameter 20 cm.

Q. 2. The area of great circle of a sphere is 100 cm².

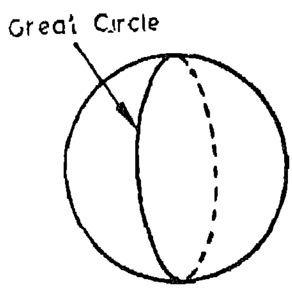



Fig. 5.67

Find its surface area.

Q. 3. Find the volume of the sphere where surface area is 80 cm².

---

*(2) Problems Involving the Volume and Surface Area of Cuboid, Cylinder, Cone and Sphere.*

**Problem 1**

A copper sphere of diameter 6 cm is drawn into a wire of 4 mm diameter. Find the length of the wire.

**Solution**

Volume of the sphere=volume of the wire.

*Note:* Wire is in the form of cylinder.

Let the length of the wire be l cm.

$$\pi (.2)^2 . l = \tfrac{4}{3} . \pi . 3^3$$

$$l = \frac{4 \times 3 \times 3 \times 3}{3 \times .2 \times .2} \text{ cm.}$$

$$= 900 \text{ cm.}$$

**Problem 2**

A sphere just fits in a cube whose volume is 1000 cc. What is the volume of the sphere ?

**Solution**

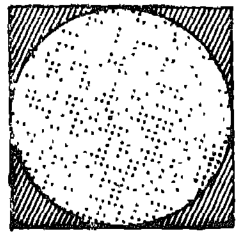


Fig. 5.68

We can see that diameter of the sphere is equal to the side of a cube.
Side of the cube $= (1000)^{\frac{1}{3}} = 10$ cm
Radius of the sphere $= 5$ cm.
Volume of the sphere $= \frac{4}{3}\pi.(5)^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 125 \text{ cc}$$

$$= \frac{11000}{21} \text{cc}$$

$$= 523\ 8 \text{ cc}$$

**Teaching Strategies**

You can collect different solids of daily use and show the radius, height, breadth, slant height etc. on these solids so that students may have a clear idea of these terms and their significance. The solids you select may be fruit like carrot or cocoanut shell, etc.

At this stage it may not be possible to give proofs of many formulae (particularly for solids with curved surfaces) as the proofs involve the use of integral calculus. It is better to take the formulae for granted (where the proofs are not possible) and show the students how to use the formulae directly or indirectly by substituting the given data.

One of the interesting activities in connection with the volume of a cone is as follows: Take a hollow cone and a cylinder with one end open and the other end closed. The cylinder shall have the height and circular base as the cone. Fill the cone with sand and pour the same in the cylinder. You will see that the cylinder will be full of sand after repeating this activity thrice. This is an activity to demonstrate (not to prove) that the volume of a cone of height 'h' and base $A = \pi r^2$ will be

$= \frac{1}{3}(\pi r^2)$ h

$= \frac{1}{3}$ (volume of the cylinder of height h and circular base of radius r.)

Sequencing of the problems according to the order in which they are taught and problems which are taught concurrently needs special attention If you look at the development of the lesson, you will observe the following sequence·

1. In the beginning we should take up those problems which involve direct use of the formulae. Students will solve such problems by substituting the given data in the formula The objective of such problems is to bring home the formula of the area and volume to the students.
2. Secondly, those problems should be taken up which involve indirect use of the formula. Here students will establish the relationship of different variables in the formula.
3. Thirdly, we can set problems which involve direct or indirect use of the formula and conversion of units. If it is a problem on cylinder, cone or sphere we can also take up problems which involve the relationship between radius and diameter along with direct or indirect use of the formula.
4. Fourthly, we should ask the students to solve problems which can be split into two sub-problems One of them is related to the volume and area and the other problem may involve the cost of labour, painting etc. Here the second part of the problem can be solved only after solving the first part because the result of the first part will be used to solve the second part.
5. Fifthly, we should give those problems to the students which can be split into two sub-problems. One of them is relating to the area and the other relating to volume or vice-versa. The objective of doing such problems is to enable the students to discriminate the formula of area and volume on the basis of dimensional unit. They will find the value of one of the variable in the first part and will use it to solve the second part.
6. Now we should take up those problems from daily life which need simultaneous use of the formula for area and volume of different regular solids.
7. Lastly, some special problems should be set up which involve the application of previous knowledge, i.e , the knowledge of other units along with the knowledge of area and volume

Students should be given enough practice in solving problems involving formulae for more than one solid so that they may be able to discriminate among the different formulae.

It should be stressed that an expression which is a product of two dimensions represents an area and similarly an expression which is a product of three dimensions represents a volume.

Q. 1. Find the volume and total surface of the following solid figure

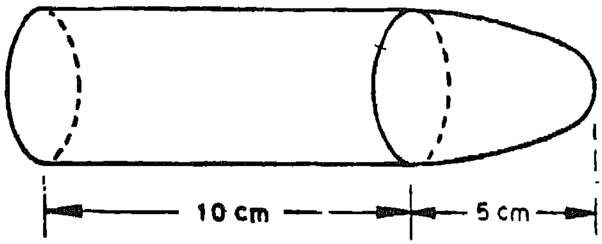



Fig. 5.69

Q. 2. A hollow spherical metal ball has an outside diameter of 25 cm and is 0.5 cm thick. Find the volume of the metal in the ball.

**Evaluation**

The main objective of this unit is to enable the student to calculate the volume and surface area of different regular solids.

**Specimen Test Items**

(1). The ratio of the volume of a cylinder to the volume of the cone on the same base and of the same height is.

(A) 1 : 3
(B) 1 . 2
(C) 2 : 1
(D) 3 : 1

(2) The volume of a spherical shell of 1 cm diameter is:
(A) $\frac{4}{3}$ cc (B) 1 cc (C) $\frac{1}{3}$ cc (D) 4 cc

(3) Total surface of a solid hemisphere of radius 1 cm is:
(A) 1 $cm^2$ (B) $\frac{1}{2}$ $cm^2$ (C) 2 $cm^2$ (D) 3 $cm^2$

(4). Find the total surface of the toy as given below :

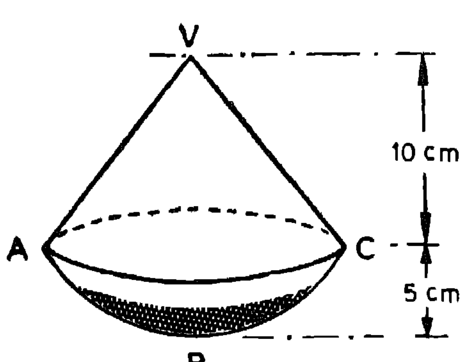



Fig. 5.70

(5). Find the area of cardboard sheet needed to prepare a rectangular open box of dimensions 12 cm $\times$ 8 cm $\times$ 6 cm

(6). If AC = 20 cm, BC = 10 cm and D is the mid-point of AB, find the volume of the frustum cut by a plane perpendicular to the axis through D

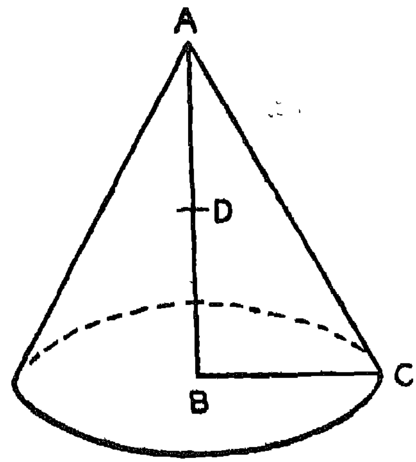



Fig. 5.71

(7). A rectangular parallelopiped is of dimensions 30 cm $\times$ 20 cm $\times$ 10 cm, it is made of wood of 1 cm thickness Find its internal volume.

(8). 100 spherical balls each of 1 cm radius are melted to form a solid cylinder of radius 10 cm. Find the height of the cylinder

**Assignment for Teachers**

(1). Collect different regular solids which are used in daily life. Draw their figures Mark their radius and length in the figures and find the volume and surface area.

(2). Prepare a chart for formula of volume and surface area of regular solids.

(3) (a) Prepare a list of teaching aids which can be used in teaching this lesson

(b) How can a mathematics museum be made useful in teaching this lesson?

(4). Construct 20 problems on areas and volumes of cuboid, cylinder, sphere and cone. Arrange them in order of difficulty level. These problems should be drawn from daily life situations.

CHAPTER 6

# Algebra

## THE CONCEPT OF A SET

### Introduction

Almost any book on "new mathematics" talks about sets and is full of symbols as $\in, \subset, \cap, \cup$, etc. There is sufficient reason for the use of these set theoretic symbols in the books of modern mathematics Set theory has become the language of modern mathematics. Without it, we not only cannot work in any branch of modern mathematics, but also we cannot even talk anything about 'new' mathematics.

In the 'traditional' mathematics, we deal with numbers and most of the work in it involves numerical computation But in modern mathematics we talk about objects other than numbers. In such discussions about non-number objects, set theory serves as an important tool and set language is a very suitable medium for such discussions.

In many modern disciplines such as sociology, linguistics, where mathematics is used for precision and clarification of concepts, set theory is widely used. So it is not out of place to have the set theory in the school curriculum so that the school students may have an elementary acquaintance with the symbols of the set theory.

It has been found by experience that school students find the concepts of a set, sub-set and super set quite confusing. So in this lesson it has been attempted to clarify these concepts and ways and methods to teach these elementary concepts have been given in great length.

### Content Covered in the Unit

1. Sets and thier elements.
   (a) Well-defined collection.
   (b) Conventions.
   (c) Representation of a set.
2. Equal and equivalent sets.
3. Finite, infinite and empty set.
4. Sub-set and universal set.

### (1) Sets and their Elements

(a) *Well-defined collection :* The terms "collection" and "element" will not have any formal definitions. You know that a week is a collec-

tion of seven days and a year is a collection of twelve months. If you are asked to write a collection of letters in English alphabet, you will write twenty six letters and there is no ambiguity. If you are asked to write the collection of all tall boys of your school, it will be ambiguous. Different meanings can be given for the word "tall".

If a collection is such that given any object, it is possible to decide whether it is in collection or not, then it is said to be well defined.

Any ambiguity or doubt in this regard will make the collection "not well defined".

Collection of interesting books, collection of wise girls and collection of beautiful women are some examples of not well-defined collections. This is because of the different meanings that can be given to the words "interesting", "wise" and "beautiful".

Collection of blind girls, collection of boys whose height is more than 150 cm and collection of Natural numbers are a few examples of well-defined collections.

In the examples given above, the collections are of similar objects. It is not necessary that this be so. We can have a collection of a table; a chair, a boy and a girl. This is also a well-defined collection.

Set : A set is a well defined collection.

**Elements of set:** The objects which form the set will be called the elements of the set.

Set will be denoted by Capital letters and the elements by small letters.

"a" is an element of "A" is denoted symbolically as

$$a \in A$$

---

Q. 1. The collection of rich boys cannot be called a set. Why ?

Q. 2 Can the collection of a number, plus sign, minus sign, a boy and a chair form a set ? Give reasons for your answer.

---

(b) *Conventions* You know that a set is denoted by writing the objects inside curly brackets. This is a **convention.**

It is also a convention that no two identical elements are listed separately as elements of a set. This convention is necessary for development of the theory of sets For example if we say {2,2,4} and {2,4} are equal, {2,2,4} has three elements and {2,4} has two elements, **thus even equal sets will not have equal number of elements.** Hence the convention.

---

Q. List the elements of the set of the letters that appear in the word "CORRELATION".

---

(c) *Representation of a Set :* There are two ways of writing sets. One is the Roster Form and the other is the Set Builder Form. Those sets, in which some property of their elements is described, are called sets written in "Set Builder Form".

Those sets, which contain a list of their elements, are called sets written in "Roster Form". Each method of writing the set has some advantage over the other, depending upon the elements of a set For example, consider the following set written in the Set Builder Form :

A= {x . x is a prime number}

To write this in the roster form is not possible because the number of prime numbers is infinite. Thus writing A in Set Builder Form is necessary. We cannot write this set in the form. {2,3,5,7,———} because it is not possible to write all the prime numbers. It may be interesting to know that there is no known formula by which a prime number can be written down.

On the other hand, consider the following set ·

B= {a table, a chair, a boy, a girl}

It can not be conveniently written in a roster form. To write this set in the Set Builder Form is possible.

B= {x · x is an element of the set consisting of a chair, a table, a boy and a girl}

But this way of writing a set is very artificial. Thus if it is possible for us to determine a common property of the elements of a set, then it is preferable to write the set in "set builder form".

Consider the set

A= {a, e, i ,o, u}

A= {x · x is a vowel}

In such cases both the forms are convenient.

---

Q. 1. Write the set of divisors of 6 in :

(a) Roster Form.

(b) Set Builder Form.

Q. 2. Write the following sets in Roster Form.

(a) X = {x : x is a two digit number divisible.}

(b) T is the set of line segments joining the P,Q,R, & S.

---

## 2. Equal and Equivalent Sets

Consider two sets :

A= {1,2,3,4}

B= {2,4,1,3}

Two sets A and B are said to be equal if they have the same elements. Thus in listing the elements of a set, writing them in different orders does not matter. The set consisting of a, b and c can be written as :

{a, b, c} , {a, c, b} , {b, a, c}

{b, c, a} , {c, a, b} , {c, b, a}

These are all equal sets.

Again consider two sets :

A= {1, 2, 3, 4}

B= {a, b, c, d}

By using one-one correspondence between their elements, it can be observed that A and B consist of the same number of elements.

Two sets A and B are said to be **equivalent** if they have the same **number of elements.**

All equal sets are equivalent sets but equivalent sets need not be equal sets.

---

Q. Which of the following pairs are equivalent sets :

(a) {T,E,A} and {T,E,A}

(b) {A,T,E} and {E,A,T}

(c) {p,q,r,s,t} and {10,11,15,200,304}

---

**3. Finite, Infinite and Empty Sets**

You have seen that sets can be written in the roster form or in the set builder form. You have seen that each form has a certain advantage over the other. Let us raise a question. Given a set, can we write them in a roster form, so that all elements are explicitly listed ? Your answer to this question will depend upon the given set. For example .

(1) If A= { x : x is a member of Rajya Sabha in India} then it is possible to write the list explicitly.

(2) If N= {x : x is a natural number} then it is impossible to write the list of the elements explicitly. However we can write N as {1,2,3,———}

A set may contain any number of elements or no elements at all We count the number of elements of a set by putting its elements in one-one correspondence with the set of counting numbers :

Let A= {p, q, r, s, t, u}

N= {1, 2, 3, 4, 5, 6———}

You say at the end of counting that there are six elements in A. Still there are natural numbers for which there is no corresponding elements in A. Thus A is not equivalent to N, and we say that A is a finite set.

Let {E=2,4,6—— }

{N=1,2,3—— }

By counting the number of elements in E by putting its elements in one-one correspondence with the counting numbers, you can say that the sets E and N are equivalent sets. The set E is infinite.

If there exists a one-one correspondence between a set X and the set N i e. if they are equivalent, then the set X is infinite.

---

Q Which of the following are finite sets ?

Also support your answer in each case

A= {x : x is a letter from English alphabets}

B= {x x is a line in a given plane}

C= {x · x is a counting number}

D= {x . x is a rational number}

E= {x : x is a multiple of 5}

---

If a set contains no element at all, it is called the **empty set** and it is denoted by the symbol $\phi$ $\phi$ is read as "phi".

An empty set is also called a null set or void set or vacuous set.

A={x : x is a natural number between 2 and 3}

Since there is no natural number between 2 & 3,

A has no element. Hence A is a null set.

Is {0} empty set ?

No, since "0" is an element of {0}.

Is {$\phi$} an empty set ?

No, since $\phi$ is an element of {$\phi$}

**4 Sub-set and universal set**

Let us consider the set of students in class IX who are coming to school by (1) cycle, (2) walking, (3) bus. Now we are forming set of students of the same class as a whole. The members of the set of students coming by cycle are the members of the set of students in class IX and so on. We say that the set of boys coming by cycle is a sub-set of the set of students of class IX

The set of boys coming to school by walking is also a sub-set of the set of students of class IX. Similarly the set of boys coming to school by bus is also a sub-set of the set of students of class IX.

In this context the set of students of class IX is called the universal set.

The set from which we select element to form a given set is called the universal set and any set formed by using some or all the elements of the universal set is called a sub-set of the universal set. In general, if A and B are any two sets such that every element of A is an element of B, we say that A is a sub-set of B and this fact is denoted by "$A \subseteq B$". If there are some elements in B which are not in A, then A is called the proper sub-set of B and is denoted by $A \subset B$.

Q. 1. Write a universal set for the following sets:

A={x : x is an integer}

B={x : x is an even number}

C={x · x is an odd number}

D={x . x is a multiple of 5}

E={x : x is a counting number}

Q 2. If U={x x is a quadrilateral} is the universal set, write any 5 sub-sets of U.

Q. 3 Is $\phi$ a sub-set of U ?

Yes, since $\phi$ has no element, we can say that all elements of $\phi$ are in U and hence it is sub-set of U.

On the basis of the above argument, we can say that $\phi$ is a sub-set of any set.

**Teaching Strategies**

*Set and its Element :* The students will be asked to give a number of examples of collections. They need not know any formal definition of collection. Taking the elements of the above collections, into consideration, emphasise that an element of the collection may be any object or idea. After having many examples of collection from the students ask them to write the collection of all tall boys in the class. Then each student may be asked to write the collection of students in the class whose heights are more than 120 cms. Compare the students' responses in the two cases. It should be pointed out that in the first case different students will come out with different collections for their answers, while in the second case the answers of all the students will be the same. Then you may ask the students the reasons for difference in their responses in the two cases. The reason for this may be explained. Finally emphasise the definition "A well-defined collection of distinct objects is called a set".

You may tell the students about the convention of using curly **braces** to represent a set. You may also explain that the elements of a set are not repeated according to convention.

*Representation of a set* : You may ask the students to list the elements of the given sets by using the well-defined property of the set. While doing so give a set in which it is not possible to list all the elements, say, the set of prime numbers.

The students will find it impossible. At this stage teach them "the set Builder form of the set". Ask them about the difficulty of writing the following type of set in Roster form ·

{x . x is the State Capitals in India}

*Equal and Equivalent set* : Ask the students to write the set {a, b, c} by changing the order of the elements. Emphasise the common property of these sets

(i) The elements of all sets are same

(ii) The number of elements of all these sets are also the same

Tell them that these sets are equal, and by changing the order of elements a set does not change.

Ask the students to compare two sets like :

{a, b, c} and {△, □, ▱}

and let them find out that in these sets the number of elements are equal. Emphasise that such sets are said to be equivalent. Give examples of two equal sets and allow the students to think whether they are equivalent. The students should conclude through examples that "Equal sets are equivalent but the converse need not be true".

*Finite and infinite set* . In the beginning the students may be asked to write the set of counting numbers as {1, 2, 3, ————}. Now the students should see the one-one correspondence between the given sets and the set of counting numbers On the basis of property of "one-one correspondence" of a set with the set of counting numbers, explain the finite and infinite set.

The students must know that in elementary cases {1, 2, 3, ————} is a tool to decide whether a given set is finite or infinite. You can tell the inquisitive students that there are infinite sets which are not in one-one correspondence with the set of counting numbers.

*Empty set* . Write any 4 sets in the set builder form and ask the students to write in roster form Out of these four sets there must be two sets in which there is no element The students will find that a set may be defined but may not contain any element. These sets are called empty sets or null sets or void sets.

The teacher may point out that {0} and {$\phi$} are *not* empty sets

*Sub-sets and universal set* : The students may be asked to form different sets of students from the set of all students of their school and name the sets as follows :

A=set of boys of your school.

B=set of girls of your school.

C=set of members of the cricket team of your school.

D=set of boys of height more than 5′-6″.

E=set of students of class IX of your school.

F=set of students of class XII of your school who offered mathematics.

and so on. This example will help you to explain universal set and subsets of universal set

Present the following universal sets : {A=x : x is a set of plane figure in a plane}

B=x x is a quadrilateral} and {C=x : x is an Indian}

and ask the students to form five sub-sets in each case At the end discuss about sub-sets of a set. Ask the students to write all sub-sets of a set consisting of 2, 3 and 4 elements. Let the students find out that the null set and the set in question are the sub-sets of the set

**Present Examples before definition of the Term-method**

If the term is introduced without adequate number of examples, the student will draw upon his pre-requisite behaviour in attempting to understand the term. In the process of figuring out just what the term does mean the student is likely to have an incorrect idea of the term Consider the two teaching strategies :

| Sequence 'A' | Sequence 'B' |
|---|---|
| (1) General statement defining a set. | (1) Example 'A' of a set so that key properties of a set may be pointed out. |
| (2) Example 'A' of a set to illustrate meaning of general statement | (2) Example 'B' of set-key properties pointed out. |
| (3) Example 'B' of a set to further illustrate meaning of general statement. | (3) Example 'C' of set-key properties pointed out |
| (4) Example 'C' etc. | (4) General statement defining a set. |

In sequence 'A', as soon as the student is given the general definition of a set, he will understand with associations based upon previous experience. The 'mental image' that the student will construct may not conform to the subsequent examples Thus, these examples should be able to correct possibly incorrect or non-conforming associations generated by giving general definition at the first instant as in sequence 'A'.

Conversely, sequence 'B' controls much of the student's "mental image" by first exposing him to example of a 'set' and pointing out the key properties of a set Thus, when the general defining statement appears in step (4) the associations and mental images of the students will very likely conform to those desired.

**Evaluation**

Some of the behavioural changes which are expected by learning this lesson are given below :

1. The student can explain the meaning of "well defined collection" by giving some examples.
2. The student can represent a set given in set builders form in Roster form and viceversa.

3 The student can discriminate between equal and equivalent sets.
4. The student can select the finite sets out of the given sets.
5. The student can write the sub-sets of a finite set
6. The student can tell that $\phi$ is the sub-set of all sets
7 The student can give examples of Universal set
8. The student can tell that every set is a sub-set of itself

**Specimen Test Items**

1. What is a set ?
2. What do you understand by an element of a set ?
3 Write the following sets in roster form
   (a) {x · x is an odd number less than 10}
   (b) {x : x is the name of an English month starting with M}
   (c) {x . x is one of the first 3 multiples of 5.}
4. Write the following sets in roster form :
   (a) Set of all counting numbers less than 20.
   (b) Set of all counting numbers greater than 2 and less than 10.
   (c) Set of all prime ministers of Bharat
   (d) Set of all Presidents of Bharat.
5 Which of the following pairs of sets are equal:
   (a) {R, C, E, A} and {A, C, E, R}
   (b) {1, 2, 3, 4} and {2, 4, 6, 8}
6. Write the number of elements in each of the following sets ·
   (a) {Shirt, Bushshirt, Pant, Pyjama}
   (b) The set of all states of Bharat.
   (c) The set of all even numbers lying between 20 and 40 (exclusive).
   (d) The set of divisors of 20.
7. Write the set of all letters which appear in the sentence 'DO NOT FORGET ME'.
8. Write the set of all digits which appear in the number 219696403.
9. Write the set {T,E,A} in all possible ways by changing the order of its elements.
10. 15 is not an element of {10, 20, 30, 40}. Write this in symbols.
11. 15 is an element of {1,3, 5,. }
    Write this in symbols.
12. Which of the following are vacuous sets ?
    (a) The set of even prime numbers.
    (b) The set of all odd numbers whose square is an even number.
    (c) The set of all prime ministers of different countries whose ages lie between 10 years and 20 years.

(d) The set of all students having passed their B.A. before they were 10 years of age.
(e) The set of black stripes in our National tricolour flag.

13. If A={1, 2, 3, 4, 5, 6, 7, 8, 9}
Write :
(a) That sub-set of A which contains all prime numbers in A.
(b) That sub-set of A which contians all even numbers in A.
(c) That sub-set of A which contains all odd numbers in A.
14. Write all sub-sets of {H, N, G}
15. State the number of all sub-sets of the following sets :
(a) {p}
(b) {p, q}
(c) {p, q, r, s}
16. Write all sub-sets of the set {p, q, r, s} having 3 elements.
17. (a) {p, q} is a sub-set of {p, q, r}.
Write in symbols.
(b) {p, q, r} is a sub-set of {p, q, r}.
Write in symbols.
(c) {p, q, r} is not a sub-set of {p, q}.
Write in symbols.
(d) The vacuous set is a sub-set of {p, q, r}.
Write in symbols

**Assignment for Teachers**

1. What is the difference between a sub-set and a proper sub-set of a set?
2. Explain how $\phi$ is the sub-set of all sets.
3. Give an example of each of the following relationships between sets :
(a) Sets A and B which are equivalent sets.
(b) Sets P and Q which are equal sets
4. Justify : The Unit on sets can be taught effectively by :
(a) Firstly presenting examples.
(b) Secondly deriving the key properties from the examples
(c) Lastly giving the definition in the form of a generalised statement.

## RELATIONS

We all in our day-to-day life experience different relationships and it is natural for mathematicians to abstract from these experiences, the

concepts of a relation and a function. A natural way to do so is to observe that for a relation to exist two elements are necessary and if we look at these elements in reverse order, then in general a different relation emerges. For example, consider two persons Mohan and Sameer. Suppose Mohan is the **father** of Sameer. Then Sameer is son of Mohan. (of course if Sudheer is a brother of Sameer, then Sameer is also **brother** of Sudheer). So it is necessary to introduce the concept of a pair of elements so that the order in which they occur is also taken into account i.e. an ordered pair. We can then form set of ordered pairs of elements of two sets A and B, the first element in the ordered pair coming from A and the second element coming from B. Thus we introduce the concept of a Cartesian product of two sets. A relation is then defined in terms of sub-sets of Cartesian product of two sets.

**Content to be covered**

(a) Cartesian product of two sets.
(b) Definition of a relation.
(c) Domain and range of a relation.
(d) Roster and Set-builder forms of a relation
(e) Types of relation.

Please follow the general instructions given below when you read this lesson .

1. We hope that you will find the material covered in this lesson easy to understand. The subject matter has been developed step by step
2 Please read each step carefully
3. Please write your answers according to the instructions given in each step at the places indicated.
4 Please check your answer with the one given at the end of the lesson If your answer does not tally with the given answer, please read the step again. If you have a doubt, please consult your teacher for clarification.
5. This is not your test, but an effort to see that you understand the concepts developed and can teach them in the classrooms. It is necessary, therefore, that you write down your answer before checking it with the given answer.
6. At the end of the lesson, there is a set of questions. Please answer all questions on separate sheets of paper and return to your teacher for evaluation

**Cartesian Product of two sets**

1 If we change the order of the elements of a set, the set does not change In what other way can you exhibit the set {a, b} ?

2. In a 2-elementic set, if we change the order of its elements, the set does not change. If we wish to emphasize the order of the elements, we write the 2-elements in parentheses ( ) instead of brackets Tick (√) those notations below in which the order of elements has been emphasized.

   (1) {2, 4} (2) {a, b} (3) {a, b} (4) {4, 2}

3. When we write 2 elements in parentheses keeping in mind their order, we say that we have written an **ordered pair**. For example, {2, 4} is an ordered pair which is quite different from the ordered pair {4,2}

   (i) Tick (√) the ordered pairs below

   (1) {1,2} (2) {a, b} (3) {3,4} (4) {3,4}

   (iii) Is the ordered pair (a,b) the same as the ordered pair (b,a) ?

   Yes/No

4. In (3,2), 3 is called the first term of the ordered pair and 2 is called the second term. What are the first and second terms of (Ram, Sita) ?

5. Two ordered pairs are equal if and only if the Ist term of one is equal to the Ist term of the other and the 2nd term of one is equal to the 2nd term of the other.

   If {x,y} = {2,3,}

   then x = . ....... . , and y = ...........

   fill in the blanks.

6. If A = {1,2} and B = {a,b}, write all possible ordered pairs, whose first term belongs to A and second term belongs to B.

   {,} {,} {,} {,}

7. If A = {2,4,6} and B = {1,5} write that set whose elements are {x,y}, where x ∈ A, y ∈ B.

8. If a∈A and b∈B, the set of all ordered pairs {a,b} is called the *cartesian product* of A and B and is denoted by A × B.

   If A = {2,4} and B = {1,3,5}, write

   A×B below

   A×B = . . .. .. .. . . ..

   . .. .. .. .. . ..

9. If A = {x,y,z}, B = {q,p}, write

   (i) A × B = .. .. .. .. .. .. ..

   .. .. .. .. .. .. ..

   (ii) State the number of elements in A × B ..... .

10. If A = {r,s,t} and B = {u,v}, then

    (i) Write A×B

    (ii) Write B × A

(iii) If A and B are different sets, which of the statements below is correct ?

(i) $A \times B = B \times A$ (ii) $A \times B \neq B \times A$

11 If $A = \{1,2\}$ write $A \times A$ below

... .......... .. ... ...... . ... ....

12. If $A = \{a,b\}$ write $A \times A$ below

. ..... ....................................

13. If $A = \{1,2,3\}$

(i) Write $A \times A$

....... ....... ... ............. ..........

(ii) Write those elements of $A \times A$ whose Ist term is equal to the 2nd term.

. ... ........... .... .... ...............

14. If $A = \{5,4,3\}$, (i) Write $A \times A$ ..............

(ii) Write the sub-set of $A \times A$ formed by those elements whose first term is greater than the second term.

............. . .......... ...

15. If $A = \{5,7\}$, $B = \{3,7,10\}$

(i) Write $A \times B$ .................

(ii) Write the sub-set of $A \times B$ formed by those elements whose Ist term is less than the IInd term.

... .. . ..................... ..............

**Relation**

16. If A= {Ram, Shyam, Mohan}

(i) Write $A \times A$

...............................

............... . ..............

(ii) The ages of Ram, Shyam, Mohan are 30, 23, 35 years respectively Write the sub-set of $A \times A$ formed by those elements in which the age of the Ist person is more than the age of the 2nd person.

............ ..... ........ ...... .

17. Smaller, bigger, taller, etc. are relations between the two terms of ordered pairs. Form ordered pairs from the set below based on the given relation.

{1,3,7}, the relation being that the Ist term is greater than the 2nd term.

{ , }, { , }, { , }

18. Form ordered pairs from the following set based on the relation given below :
{Jawaharlal Nehru, Indira Gandhi, Lal Bahadur Shastri}
The first person was the Prime Minister before the second person.
... .......... ... ...................

19. If A = {4,8,12}
(i) Write $A \times A$ ........
(ii) All ordered pairs formed from the set {4,8,12} are present/are not present in $A \times A$.

20. If $r_1 = \{(1,1), (2,2)\}$, and
$r_2 = \{(1,2), (1,1)\ (2,1), (2,2)\}$ tick (√) the correct statements below :

(i) $r_1 \subset r_2$ (iv) $(2,1) \notin r_2$
(ii) $r_1 \subseteq r_2$ (v) $(2,1) \in r_1$
(iii) $r_1 \not\subset r_2$

21. The set of ordered pairs formed from set A on the basis of some relation (is a sub-set/is not a sub-set) of $A \times A$. Choose a phrase from the bracket to make a correct statement.

22. .......... . ... ........... .....
If A = {3,6,9}
(i) Write $A \times A$
...... .......... ....... ......
...... .......... ..................
(ii) Write that subset of $A \times A$ in whose elements the Ist term is a factor of the 2nd term.

23. *Definition* : Any relation from a set B is a subset of $A \times B$, formed by the elements (a,b) where
(i) $a \in A$, $b \in B$
(ii) a and b are related by the relation r

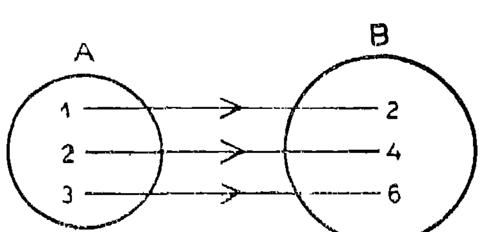



Fig. 6.1

In the above figure the relation 'r' between two sets A and B is shown by arrows.

Answer the following questions :

(i) What is the relation between A and B ?

(ii) Write r= { }, (iii) Write $A \times B$

24. In question 23

(1) Are all elements of r also elements of $A \times B$ ? Yes/No

(2) Is r a sub set of $A \times B$ ? Yes/No

25. Any relation r from A to B is that sub-set of $A \times B$, for which whenever (a,b) is an element of r, the relation r exists between a and b

If $A = \{4,5\}$

$B = \{16, 20, 30\}$

Write that relation whose elements (a, b) are such that a is a factor of b

r = ... ...... . ........

26. Write the relation r from A to B, such that $(a, b) \in r$, if and only if, a=b

$A = \{2,4,5\}$, $B = \{2,3,4\}$

r = ............... ..... . ...... .

27. If $A = \{2,3,4,5\}$

$B = \{2,3,4\}$

Write the relation r from A to B, for each of whose elements (a,b) a should be greater than b.

r = ...... . ........... ...... .

28. (i) Any relation r from A to B is a ... .. of $A \times B$.

(ii) Any sub-set of $A \times B$ can be called a ...... from A to B.

29. If **Domain and Range** (a, b) is any element of some relation r from A to B tick (√) the correct statements below .

(1) $a \in A$ (2) $b \in A$ (3) $a \in B$ (4) $b \in B$

30. If r is a relation from A to B, where

$r = \{(1,1), (1,b), (3,a), (2,c), (2,d)\}$ tick (√) the correct statement(s) below ·

(i) $\{1,2,3\} \subset A$ (iv) $\{a,b,c,d\} \subset A$

(ii) $\{1,2,3,\} \subset B$ (v) $\{1,a,b\} \subset B$

(iii) $\{a,b,c,d\} \subset A$ (vi) $\{1, a, b\} \subset B$

31. If $A = \{a,b,c\}$, $B = \{x,y,z,u\}$, and r is a relation from A to B, *tick (√) the correct statements below.*

(i) The set formed by taking all first terms of the elements of **r** as its elements is a sub-set of A.

(ii) The set formed by taking all 2nd terms of the ordered pairs, which are the elements of r, as its elements, is a sub-set of B.

(iii) $r \subset A \times B$

32. The set formed by the 1st terms of all ordered pairs, which are elements of a relation r, is called the *domain* of r, and the set formed by the 2nd terms of these ordered pairs is called the *range* of r.

If $r = (\pi, 3.14), (e, 2.16), (\sqrt{2}, 1.41)$

Write the domain and range of r

Domain = { }

Range = { }

33. If $r = \{(18,16), (8, 15), (8,14), (8,13)\}$

domain of r =

range of r =

34. If $A = \{2,3,5\}$

$B = \{5,10\}$

and there exists a relation r from A to B, where $r = \{(2,6), (2,10), (3,6), (5,10)\}$ state, if $(a,b) \in r$, which of the following statements is true for (a,b) ?

(i) a is less than b (ii) a is a factor of b.

**Roster and Set-builder forms of a Relation**

35. If $A = \{a, b, c, d\}$ and $B = \{p, q, r\}$ tick (√) those relations below, whose domain is a sub-set of A and range is a sub-set of **B**.

(i) $\{(a, p), (a, q), (d, r)\}$

(ii) $\{(a, p), (b, q), (c, r), (d, r)\}$

(iii) $\{(p, a), (p, d), (q, c), (r, d)\}$

36. If $x, y \in N$, where $N = \{1, 2, 3 - - -\}$ then the relation r, where $(x,y) \in r$ if and only if $y = 2x$ can be exhibited in the following 2 ways :

(a) $\{(1, 2), (2, 4), (3, 6) - - -\}$

(b) $\{(x, y) : x, y \in N \text{ and } y = 2x\}$

The form (b) of the relation r is called the **set builder form of r**

If $r = \{(1, 1), (2, 2), (3, 3) - - -\}$

Write it in the set builder form.

r = ... ........... .. .. . .

37. If $x, y \in N$, where $N = \{1, 2, 3 ———\}$ formulate correct pairs of the following relations expressed in roster and set builder forms.

(i) $\{ (1, 3), \quad (2, 6), \quad (3, 9), ———\}$

(ii) $\{ (1, 4), \quad (2, 5), \quad (3, 6), ———\}$

(iii) $\{ (1, 3), \quad (2, 5), \quad (3, 7), ———\}$

(iv) $\{ (1, 3), \quad (2, 4), \quad (3, 5), ———\}$

(a) $\{ (x, y) : x, y \in N \text{ and } y = x + 2 \}$

(b) $\{ (x, y) : x, y \in N \text{ and } y = 2x+1 \}$

(c) $\{ (x, y) \cdot x, y \in N \text{ and } y = x+3 \}$

(d) $\{ (x, y) \cdot x, y \in N \text{ and } y = 3x \}$

(e) $\{ (x, y) \cdot x, y \in N \text{ and } x = y+4\}$

(i) .. . .... (d) (ii) ............. ( )

(iii) .. .. . . . ( ) (iv) .............. ( )

38. If $x, y \in N$, where $N = \{1, 2, 3, \ldots \ldots \ldots\}$ write the relation r in roster form, where $r = \{ (x, y) : x=2y\}$, $r = \{ (2, 1), (\ldots 2), (\ldots 3), \ldots\ldots\ldots .\}$

39. If $x, y \in N$, write the following relation r in roster form.

$r = \{ (x, y) : y = 3x+1\}$

r =

40. If $r = \{ (1, 1), \quad (2, 4), \quad (3, 9), \quad (4, 16) ————\}$ write it in set builder form.

**Inverse Relation**

Given a relation $r_1 = \{ (a, b) : a, b \in A\}$

The relation $r_2 = \{ (b, a) : (a, b) \in r_1\}$

is called the inverse relation of $r_1$

41. Suppose that $(a, b) \in r_1$ if and only if $(b, a) \in r_2$

Then write $r_2$=

42 Write the inverse relations of the relations given below :

(i) $\{ (a, b), \quad (h, n), \quad (k, g) \}$

Inverse relation = — — —

(ii) $\{ (2, 6), \quad (9, 12), \quad (12, 18) \}$

Inverse relation = — — —

43. The relation inverse to a given relation r is denoted by $r^{-1}$
If r = (5, 10), (2, 10), (3, 20) write $r^{-1}$
$r^{-1}$= — — — —

44. If $r = \{ (x, y) : x = 2y\}$, write $r^{-1}$
$r^{-1}$ =

45. If $r = \{ (1, 2), (3, 5), (5, 10)\}$ and
$r^{-1} = \{(2, 1)\ (5, 3), (10, 5) \}$
then $r^{-1}$ is the ................. relation of r.

46. If $S = \{x : x \text{ is a student}\}$
and $r = \{ (x, y) \in S \times S \cdot x \text{ studies in the institution of } y\}$
answer the following questions .
(i) If $(y, z) \in r$, what is the relation between y and z ?
(ii) If $x \in S$, what does $(x, x) \in r$ mean ?
(iii) For every $x \in S$, is $(x, x)$ an element of r ? (Yes/No)

**Types of Relation**

47. If A = {1, 2, 3, 4}
and $r = \{ (x, y) \in A \times A : x<y\}$
answer the following questions:
(i) If $(y, z) \in r$, how is y related to z.
(ii) If $x \in A$, state with reason, whether $(x, x) \in r$ ?

48. If A = {1, 2, 3, 4, 5, 6}
$r = \{ (x, y) \in A \times A : x \text{ divides } y\}$
answer the following questions :
(i) Does every element of A divide itself ? (Yes/No)
(ii) Write r in roster form.
.........................
(iii) Write those elements of r which exhibit the fact that every element of $x \in A$ divides itself.

49. If A = {1, 2, 3 — — —} tick (√) those relations from A to A given below, in which every element of A is related also to itself under the given relation.
$r_1 = \{ (x, y) \in A \times A : x \text{ divides } y\}$
$r_2 = \{ (x, y) \in A \times A : x > y\}$
$r_3 = \{ (x, y) \in A \times A : x = y\}$
$r_4 = \{ (x, y) \in A \times A : x \leqslant y\}$

50. For all relations from A to A, in which every element of A is related also to itself under the given relation, tick (√) the correct statement(s) below :

(i) $(x, x) \in r$, for every element $x \in A$.

(ii) $(x, x) \notin r$, for every element $x \in A$.

(iii) $(x, x) \in A$, for some elements x, but not all, belong to A.

51. Any relation r from A to A, which has the property that every element $x \in A$ is related to itself, is called **a reflexive relation.** Tick (√) such relations below :

(1) $A = \{x : x \text{ is a student}\}$
$r = \{(x, y) \in A \times A : x \text{ studies in the same class as } y\}$

(2) $A = \{x : x \text{ is a straight line}\}$
$r = \{(x, y) \in A \times A : x \text{ is perpendicular to } y\}$

(3) $A = \{x : x \text{ is a citizen}\}$
$r = \{(x, y) \in A \times A : x \text{ lives in the city of } y\}$

(4) $A = \{2, 4, 6, ---\}$
$r = \{(x, y) \in A \times A : x \text{ divides } y\}$

52. Which of the following properties must a reflexive relation have ?

(i) Every element of A must be related to itself under the given relation.

(ii) Some elements of A, but not all, must be related also to themselves under the given relation.

(iii) All elements of r should be of the form (x, x) i. e. the 1st term and the 2nd term of every element of r must be the same.

(iv) The number of elements of r of the form (x, x) must be the same as the number of elements in A.

53 If r is reflexive relation from A to A, what property must r have ?

.....................................

.....................................

54. If r is a relation from A to A, such that every element of A is related also to itself, what type of relation is r called ?

55. If r is a reflexive relation from A to A, tick (√) the correct statement(s) below :

(i) For every element $x \in A$, $(x, x) \in r$

(ii) For some elements x, but not all, belonging to A
$(x, x) \in r$.

(iii) For every element $x \in A$, $(x, x) \ni r$.

56. Every triangle is similar to itself. If $A = \{x : x \text{ is a triangle in a given plane}\}$ and $r = \{(x, y) \in A \times A : x \text{ is similar to } y\}$, is r a reflexive relation ? (Yes/No)

57. If $A = \{x : x \text{ is a straight line in a given plane}\}$
$r = \{(x, y) \in A \times A : x \text{ is perpendicular to } y\}$
Answer the following questions :
(i) If line x is perpendicular to line y, will line y be perpendicular to line x ? (Yes/No)
(ii) If $(x, y) \in r$, does $(y, x) \in r$ ? (Yes/No)
(iii) Does $(x, x) \in r$ ? (Yes/No)
(iv) Is the relation r reflexive ? (Yes/No)

58. Tick (√) those relations below for which $(x, y) \in r$ implies that $(y, x)$ also $\in r$.
(1) $A = \{ x \cdot x \text{ is a triangle in a given plane}\}$
$r = \{ (x, y) \quad A \times A : x \text{ is similar to } y\}$
(2) $A = \{ x . x \text{ is a student residing in a given hostel}\}$
$r = \{ (x, y) \in A \times A \cdot x \text{ lives in front of } y\}$
(3) $A = \{x . x \text{ is a straight line in a given plane}\}$
$r = \{(x, y) \; A \times A : x \text{ is parallel to } y\}$
(4) $A = \{x : x \text{ is a real number}\}$
$r = \{(x, y) : x \text{ is greater than } y\}$

59. If $A = \{1, 2\}$ and $r_1 = \{(1, 2), \; (1, 1), \; (2, 2) \; (2, 1)\}$ while $r_2 = \{(1, 2), (1, 1), (2, 2)\}$; answer the following questions :
(1) If $(x, y) \in r_1$ is it necessary that $(y, x) \in r_1$ ? (Yes/No)
(2) Does $(x, y) \in r_2$ imply that $(y, x) \in r_2$ ? (Yes/No)

60. Those relations in which $(x, y) \in r$ implies that $(y,x) \in r$, are called symmetric. Tick (√) the symmetric relations below :
(1) $A = \{1, 2, 3\}$
and $r = \{(1, 3), \; (3, 1), \; (2, 3), \; (3, 2), (3, 3)\}$
(2) $A = \{1, 2, 3, ---\}$
and $r = \{ (x, y) \in A \times A : x \text{ is less than } y\}$
(3) $A = \{x . x \text{ is a boy}\}$
and $r = \{(x, y) \in A \times A : x \text{ is the brother of } y\}$

61. Tick (√) those relations below which are both reflexive and symmetric.
(1) $A = \{1, 2, 3\}$
and $r = \{(1, 2), \; (1, 3), \; (1, 1), \; (2, 2), \; (3, 3), \; (2, 1), \; (3, 1)\}$

(2) $A = \{1, 2, 3, 4\}$
and $r = \{(1, 2), \ (2, 1), \ (3, 3), \ (4, 4)\}$
(3) $A = \{x : x$ is a straight line in a given plane$\}$
and $r = \{(x, y) \in A \times A : x$ is perpendicular to $y\}$
(4) $A = \{x : x$ is a boy$\}$
and $r = \{(x, y) \in A \times A : x$ has the colour of $y\}$

62. Write those relations of the above question no. 61 which are symmetric but not reflexive.

63 If $A = \{x, y, z\}$ and $r = \{(x, y), \ (y, x), \ (x, x,) \ (y, z), \ (z, y)\}$
(1) Is r a symmetric relation ? (Yes/No)
(2) Write the inverse relation of r.
— — — — — — — — — —
(3) Is the inverse relation of r also r ? (Yes/No)

64. Tick (√) those properties below which are necessary for a relation r to be symmetric.
(1) $(x, y) \in r$ implies $(y, x) \in r$.
(2) $r^{-1} = r$
(3) $(x, x) \in r$ for every element x
(4) The relation r is identical with its inverse relation $r^{-1}$.

65. Match the following :
(1) $(x, x) \in r$ for every element $x \in A$ (a) Reflexive relation.
(2) $(x, y) \in r$ implies that $(y, x) \in r$ (b) Symmetric relation.

66 If $A = \{1, 2, 3, ———\}$
$r = \{(x, y) \in A \times A : x < y\}$
answer the following questions .
(1) If $x < y$, $y < z$, is it necessary $x < z$ ? (Yes/No)
(2) If $(x, y) \in r$ and $(y, z) \in r$, will $(x, z)$ also necessarily $\in r$ ? (Yes/No)
(3) Is the relation r reflexive ? (Yes/No)
(4) Is the relation r symmetric ? (Yes/No)

67. Tick (√) those relations below for which $(x, y) \in r$ and $(y, z) \in r$ both together imply that $(x, z) \in r$.
(1) $A = \{x : x$ is a student$\}$
$r = \{(x, y) \in A \times A : x$ studies in the institution of $y\}$
(2) $A = \{x \cdot x$ is a straight line in a given plane$\}$
$r = \{(x\ y) \cup A \times A : x$ is parallel to $y\}$

(3) $A = \{x : x$ is a straight line in the given Cartesian plane$\}$
$r = \{(x, y) \in A \times A : x$ has the same slope as $y\}$

(4) $A = \{x \cdot x$ is a straight line in a given plane$\}$
$r = \{(x, y) \in A \times A \cdot x$ is perpendicular to $y\}$

(5) $A = \{x : x$ is a boy$\}$
$r = \{(x, y) \in A \times A : x$ is a brother of $y\}$

(6) $A = \{x : x$ is a boy$\}$
$r = \{(x, y) \in A \times A : x$ has the same father and mother as $y\}$

68. If $A = \{1, 2, 3\}$

$r_1 = \{(1, 2), \ (2, 3), \ (1, 3), \ (3, 2), (2, 2)\}$

Answer the following questions :

(1) If $(x, y)$ and $(y, z) \in r_1$, is it necessarily true that $(x, z) \in r_1$ ? (Yes/No)

(2) If not, what ordered pairs should be included in $r_1$ so that the above implication may become true ?

.. . ......................

Answer the same questions foi the following relation $r_2$ also where :

$r_2 = \{(1, 2), \ (2, 3), \ (1, 3), \ (3,2)\}$ (Yes/No)

(3) — — — — — — — — —

(4) — — — — — — — — —

69. Those relations, in which if $(x, y) \in r$ and $(y, z) \in r$, then also $(x, z)$ necessarily $\in r$, are called *transitive.* Tick ($\surd$) the transitive relations below ·

(1) $A = \{1, 2, 3, 4\}$

and $r = \{(2, 3), \ (3, 4), \ (2, 4)\}$

(2) $A = \{1, 2, 3\}$

and $r = \{(x, y) \in A \times A : x$ is the mother of $y\}$

(3) $A = \{x : x$ is a girl or lady$\}$

and $r = \{(x, y) \in A \times A . x$ is the mother of $y\}$

70. If $A = \{x :$ is a person$\}$, look at the following relations from A to A and place ($\surd$) at the correct places and ($\times$) at the incorrect places in the table below .

$r_1 = \{(x, y) \in A \times A :$ the age of x is less than that of $y\}$

$r_2 = \{(x, y) \in A \times A : x$ lives in the house adjacent to that of $y\}$

$r_3 = \{(x, y) \in A \times A : x$ lives in the same mohalla as $y\}$

$r_4 = \{(x, y) \in A \times A \cdot x$ is the father of $y\}$

$r_5 = \{(x, y) \in A \times A : x \text{ is a brother of } y\}$
$r_6 — \{(x, y) \in A \times A \cdot x \text{ has the same father and mother as } y\}$
$r_7 = \{(x, y) \in A \times A : x \text{ lives in front of } y\}$

| | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ |
|---|---|---|---|---|---|---|---|
| Reflexive | X | | | | | | |
| Symmetric | X | | | | | | |
| Transitive | √ | | | | | | |

71. If $r_1, r_2, r_3, r_4, r_5, r_6, r_7$ are the above relations, which of them are :
   (1) Reflexive but neither symmetric nor transitive.
   (2) Symmetric but neither reflexive nor transitive.
   (3) Transitive but neither reflexive nor symmetric
      (i) ..................
      (ii) ................
      (iii) ....... ........

72. Write which of the above relation(s) is/are reflexive, symmetric and transitive

   .. ........... ......

73. Those relations which are reflexive, symmetric and transitive are called **equivalence** relations.

   If $A = \{........., -3, -2, -1, 0, 1,2,3, .........\}$
   $r = \{ (x,y) \in A \times, A : 6 \text{ divides } x—y\}$
   Is r an equivalence relation ? (Yes/No)

   Tick (√) the correct statements below :

   | | |
   |---|---|
   | (1) $(15,27) \in r$ | (2) $(27,15) \in r$ |
   | (3) $(38,38) \in r$ | (4) $(17,16) \in r$ |
   | (5) $(-6,6) \in r$ | (6) $(-10,10) \in r$ |
   | (7) $(4, -8) \in r$ | (8) $(2,4) \in r$ |

74. If r is a relation from $A \in$ to A, and if $a,b,c, \in A$, then match the following :
   (1) Reflexive (i) $(a,a) \in r$ for all $a \in A$
   (2) Symmetric (ii) $(a,b) \in r$ implies that $(b,a) \in r$
   (3) Transitive (iii) $(a,b) \in r$ and $(b,c) \in r$ together imply that $(a,c) \in r$

## ANSWERS

1. {b, a}
2. (3) (4)
3. (i) (1) (2) (4)
   (ii) No, unless $a = b$.
4. First term · Ram
   Second term : Sita
5. $x = 2$, $y = 3$.
6. (1, a), (1, b), (2, a), (2, b).
7. {(2, 1), (2, 5), (4, 1), (4, 5), (6, 1), (6, 5)}
8. {(2, 1), (2, 3), (2, 5), (4, 1), (4, 3), (4, 5)}
9. (i) $A \times B$ = {(x, p), (x, q), (y, p), (y, q), (z, p), (z, q)}
   (ii) 6
10. (i) {(r, u), (r, v), (s, u), (s, v), (t, u), (t, v)}
    (ii) {(u, r), (v, r), (u, s), (v, s), (u, t), (v, t)}
    (iii) $A \times B \neq B \times A$
11. {(1, 1), (1, 2), (2, 1), (2, 2)}
12. {(a, s), (a, b), (b, a), (b, b)}
13. (i) {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
    (ii) {(1, 1), (2, 2), (3, 3)}
14. (i) {(5, 5), (5, 4), (5, 3), (4, 5), (4, 4), (4, 3), (3, 5), (3, 4), (3, 3)}
    (ii) {(5, 4), (5, 3), (4, 3)}
15. (i) {(5, 3), (5, 7), (5, 10), (7, 3), (7, 7), (7, 10)}
    (ii) {(5, 7), (5, 10), (7, 10)}
16. (i) {(Ram, Ram), (Ram, Shyam), (Ram, Mohan), (Shyam, Ram) (Mohan, Ram), (Shyam, Shyam), (Shyam, Mohan), (Mohan, Shyam), (Mohan, Mohan)}
    (ii) {(Ram, Shyam), (Mohan, Ram), (Mohan, Shyam)}
17. (3, 1), (7, 1), (7, 3).
18. (Jawaharlal Nehru, Indira Gandhi),
    (Jawaharlal Nehru, Lal Bahadur Shastri)
    (Lal Bahadur Shastri, Indira Gandhi)
19. (i) {(4, 4), (4, 8), (4, 12), (8, 4)), (8, 8), (8, 12), (12, 4), (12, 8), (12, 12)}
    (ii) Are present
20. (i) (iv) (v)

21. is a subset
22. (i) (3, 3), (3, 6) (3, 9), (6, 3), (6, 6), (6, 9), (9, 3), (9, 6), (9, 9)
 (ii) (3, 3), (3, 6), (3, 9), (6, 6), (9, 9)
23. (i) Elements of A are half the corresponding elements of B.
 (ii) (1, 2), (2, 4), (3, 6).
 (iii) r = (1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 2), (3, 4), (3, 6)
24. (1) Yes (2) Yes
25. r = (4, 16), (4, 20), (5, 20), (5, 30)
26. r = (2, 2), (4, 4)
27. r = (4, 2) (4, 3), (5, 2), (5, 3), (5, 4)
28. (i) Subset (ii) Relation
29. (1) (4)
30. (i) (iv)
31. (i) (ii) (iii)
32. Domain = { e, 2 }
 Range = { 3.14, 2.16, 1.41 }
33. Domain = { 18, 8 }
 Range = { 16, 15, 14, 13 }
34. (i) (ii)
35. (i) (ii)
36. $r = \{(x, y) ; x, y. N, \text{ and } y = x\}$
37. (ii)... (c) (iii)... (b) (iv)...(a)
38. $r = \{(2, 1), (4, 2), (6, 3), \ldots\ldots\ldots\}$
39. $r = \{(1, 4), (2, 7), (3, 10) \ldots\ldots\ldots\}$
40. $r = \{(x, y) , x, y \text{ N and } y = x^2\}$
41. $r_2 = \{(4, 3), (6, 5), (8, 7)\}$
42. (i) {(b, a), (n, h), (g, k)}
 (ii) {(6, 2), (12, 9), (18, 12)}
43. $r^{-1} = \{(10, 5), (10, 2), (20, 3)\}$
44. $r^{-1} = \{(y, x) : y = x/2\}$
45. Inverse
46. (i) y studies in the institution of z (ii) x studies in the institution of x or y and z study at the same institution. (iii) Yes.
47. (i) $y < z$
 (ii) No.
 Because x can never be less than x.

48. (i) Yes
    (ii) {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2) (2, 4), (2, 6), (3, 3) (3, 6), (4, 4), (5, 5) (6, 6)}
    (iii) {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}
49. $R_1$, $R_3$ $R_4$
50. (i)
51. (1) (3) (4)
52. (i) (iv)
53. Every element of A must be r—related also to itself.
54. Reflexive
55. (i)
56. Yes
57. (i) Yes (ii) Yes (iii) No (iv) No
58. (1) (2) (3)
59. (i) Yes (ii) No
60. (1) (3)
61. (1) (4)
62 (2) (3)
63. (i) Yes
    (ii) {(y, x), (x, y), (x, x), (z, y), (y, z)}
    (iii) Yes
64. (i) (ii) (iv)
65. (1)......(a) (2)......(b)
66. (i) Yes (ii) Yes (iii) No (iv) No.
67. (1) (3) (6)
68. (1) No. (2) (3, 3) (3) No (4) (2, 2), (3, 3)
69. (1) (2)
70.

| | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ |
|---|---|---|---|---|---|---|---|
| Reflexive | × | × | √ | × | × | √ | × |
| Symmetric | × | √ | √ | × | × | √ | √ |
| Transitive | √ | × | √ | × | × | √ | × |

71. (1). ...(None)
    (2) .....($r_2$, $r_7$)
    (3) .....($r_1$)

72. $r_3$, $r_6$
73. Yes (1), (2), (3), (5), (7)
74. (1)......(i)
    (2) ...(ii)
    (3)....(iii)

**Teaching Strategies**

This lesson is prepared for you on the lines of programmed instructions. It designs the teaching strategy as follows ·

(A) Specify, in advance, in behavioural terms what the learner is expected to do at the end of the instruction.

(B) Specify what the learner is able to do at the beginning of the instruction.

(C) Arrange the sequence of tasks starting from the entering behaviour and leading the students effectively towards the terminal behaviour using the following principles of learning.

(*i*) *Principle of Small Steps*

The principle states that the student can learn better if he learns in small steps. Topic is presented in small steps. Each small step consists of a part of a terminal behaviour and is called a frame

(*ii*) *Principle of Active Responding*

The principle ensures that the student will learn better if he participates actively in the lesson. In programmed instruction each student has to attempt every frame and write the answer in the space provided in the frame.

(*iii*) *Principle of Immediate Confirmation*

The student will be reinforced if he gets immediate confirmation of his progress. In programmed instruction, after giving response the student will like to confirm his answer. If the confirmation is immediate, the student will progress and move on to the next frame. Answer is arranged in such a way that the student can see it after writing his response.

(*iv*) *Principle of Self-pacing*

A student will learn according to his intelligence and ability In programmed instruction, the student is not compelled to complete all

the frames of a programme in one or two sittings. The student goes through each frame taking his own time, which depends upon his success in completing the frames.

(v) *Principle of Student Testing*

Students are the best judge of the teaching of a teacher. The student response is considered as an integral part of the learning process. The programme is tried out over a number of students before taking it to the field. The defects in the frames will be pointed out by the students during try out and the same can be corrected

*Evaluation*

Some of the behavioural changes which are expected by learning this lesson are given below :

1 Student can write the product of two sets.
2. Student can define relation from a set A to the Set B.
3. Student can find out the given relation from the set A to B.
4. Student can discriminate between different types of relations.

**Specimen Test Items**

1. If $A = \{1,2,3\}$, $B = \{3,4,5\}$, $C = \{4,5,6,7\}$
   Find the following Cartesian products:
   (a) $A \times C$
   (b) $B \times C$
   (c) $C \times A$
   (d) $A \times A$
   (e) $C \times B$
2. If $(3, 4) \in A \times B$, which of the following must always be true?
   (1) $3 \in A$, $4 \in B$
   (2) $3 \in B$, $4 \in A$
   (3) $3 \in A$, $4 \in A$
   (4) $4 \in A$, $4 \in B$
3. If $A = \{h,n,g\}$ and $B = \{r,c,e,a\}$, which of the following relations is not from A to B
   (a) $\{(h,r), (n,r), (g,r)\}$
   (b) $\{(h,a), (h,e), (h,c), (n,c), (n,e), (n,a)\}$
   (c) $\{(n,c), (e,n), (n,e), (h,a), (g,r)\}$
   (d) $\{(h,a), (n,c), (g,c)\}$

4. If the relation $r = \{(5,6), (6,5), (4,4), (3,2)\}$ the domain of r will be:

   A $\{2,3,4,5,7\}$

   B $\{3,4,5,6\}$

   C $\{2,4,5,6\}$

   D $\{4,5,6\}$

5. If $r = \{(1,2), (2,4), (2,3)\}$ the inverse relation $r^{-1}$ of r will be:

   A $\{(1,2), (2,4), (2,3)\}$

   B $\{(2,1), (2,4), (2,3)\}$

   C $\{(2,1), (4,2), (3,2)\}$

   D $\{(2,1), (4,2), (2,3)\}$

6. If $A = \{k,d,g\}$ and $B = \{p,q,a\}$, write the relation r in roster form, where·

   $r = \{(x,y) \in A \times B :$ x comes in the alphabet before y$\}$

7. If $A = \{1,2,3,4\}$ write the following relation r in roster form.
   $r = \{(x,y) \in A \times A$ . x is less than y)$\}$

8. If $A = \{x$· x is a person$\}$, which of the relation below is reflexive and transitive but not symmetric?

   (A) $\{(x,y) \in A \times A$ · the age of x is more than that of y$\}$

   (B) $\{(x,y) \in A \times A$ : x lives in the city of y)$\}$

   (C) $\{(x,y) \in A \times A$ : x lives in a storey directly above y)$\}$

   (D) $\{(x,y) \in A \times A$ : x is a sister of y$\}$

   (E) $\{(x,y) \in A \times A$ · x is either y or a sister of y$\}$

9. If $A = \{1,2,3,4\}$ point out that relation below which is symmetric but neither reflexive nor transitive.

   (A) $\{(1,1), (2,2), (3,3), (1,3), (2,3), (4,4), (2,4)\}$

   (B) $\{(1,1)\ (2,2), (3,3), (4,4), (2,3), (3,2)\}$

   (C) $\{(2,3), (3,2), (1,3), (3.1)\}$

   (D) $\{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

10. If $A = \{x,y,z\}$, which of the following relations is an equivalence relation?

    (A) $\{(x,x), (x,y), (y,y), (y,z), (x,z), (z,z)\}$

    (B) $\{(x,x), (x,y), (y,y), (z,z), (x,z), (z,x)\}$

    (C) $\{(x,x), (y,y), (z,z), (z,y), (y,x), (x,z), (z,x)\}$

    (D) $\{(x,x), (y,y), (z,z), (x,y), (y,x), (x,z), (z,x), (y,z), (z,y)\}$

11. If $(x,y) \in r$ and $(y,z) \in r$ both together imply that $(x,z) \in r$, such a relation is:
    (A) Reflexive
    (B) Symmetric
    (C) Transitive
    (D) Equivalence.

12. If r is a symmetric relation which of the following statements is necessarily true? The relation is from A to A:
    (A) $(x,x) \in r$ for every $x \in A$
    (B) $(x,y) \in r$ implies that $(y,x) \in r$ for every $x,y \in A$
    (C) $(x,y) \in r$ and $(y,z) \in r$ both together imply that for every $x,y,z \in A$
    (D) $(x,x) \in r$ for some elements x, but not all, belonging to A.

**Assignment for Teachers**

1. What do you understand by relation?
2. Explain by giving example what you understand by the inverse relation of a relation.
3. Give an example of a relation which is reflexive but neither symmetric nor transitive.
4. Give an example of a relation which is symmetric but neither reflexive nor transitive.
5. Give an example of a relation which is transitive but neither reflexive nor symmetric.
6. Prepare a programmed lesson for teaching the concept of a function.

## FUNCTIONS AND GRAPHS

**Introduction**

The word "function" means 'operation'. For example, when we say that 2x is a function of x, we mean that we get a new number 2x after multiplying (performing the operation of multiplication) the given number x by 2. As you can see in this example, the value of 2 x depends upon that of x. In common parlance a mathematical statement such as $y=f(x)$, where the value of y depends on that of x means that "y is a function of x"

But this understanding of a function creates difficulties in many mathematical discussions For example, if $y = \pm \sqrt{x}$, we have generally two values of y corresponding to a given value of x. Now the question arises —which value to choose for our discussion. So keeping this difficulty in

view, we modify the definition of a function of x so as to include only single-valued functions of x.

Also the concept of a function of x is needed very much for many branches of higher mathematics such as topology and functional analysis wherein the concepts are developed in terms of set theory. So for use in higher mathematics a set-theoretic definition of function is also needed. In this lesson we will start with the set-theoretic definition of a function, but will mostly concentrate on the graphs of functions which are so useful in mathematical discussion or otherwise.

**Content Covered in this Lesson**

(1) Definition of a function
(2) Graphs of functions

**Development of the Content**

We start with the following set-theoretic definition of a function.

*Definition*: Let X and Y be two non-empty sets. A function f from X to Y, written as $f : X \rightarrow Y$ is a sub-set of $X \times Y$, such that no two different ordered pairs in the set f have the same first entry, that is, if $(x,y) \in f$ and $(x, y_2) \in f$ imply $y_1 = y_2$. Further, for every $x \in X$, there exists a $y \in Y$ such that $(x,y) \in f$

The set of all first entries in the ordered pairs (x,y) of the function of f is called the domain of f and the set of all second entries in same ordered pairs is called the range of f. The variable x taking values in the domain is often called the independent variable and the variable y taking values on the range is called the dependent variable.

From the definition of a function we can conclude that a function f from X to Y is a correspondence which associates with *each* element x of X a *unique* element y of Y. This fact is a useful tool to examine whether a certain relation is a function. Diagrammatic representations of some selections which are examples and non-examples of a function are given in Figs. 6.2 to 6.7 on page 224 to clarify the concept of a function and when a relation is not a function.

Generally, at the school level the sets X and Y connected with a function are sub-set of R, the set of all real numbers. Such functions are called functions of a real variable or real-valued functions of a real variable.

We know that in the Cartesian plane any ordered pair of numbers can be represented by a point (x,y) —x along OX or the x—axis and y along OY or the y-axis. So the set of ordered pairs of a function of a real variable has a representation in the Cartesian plane, and this set of points

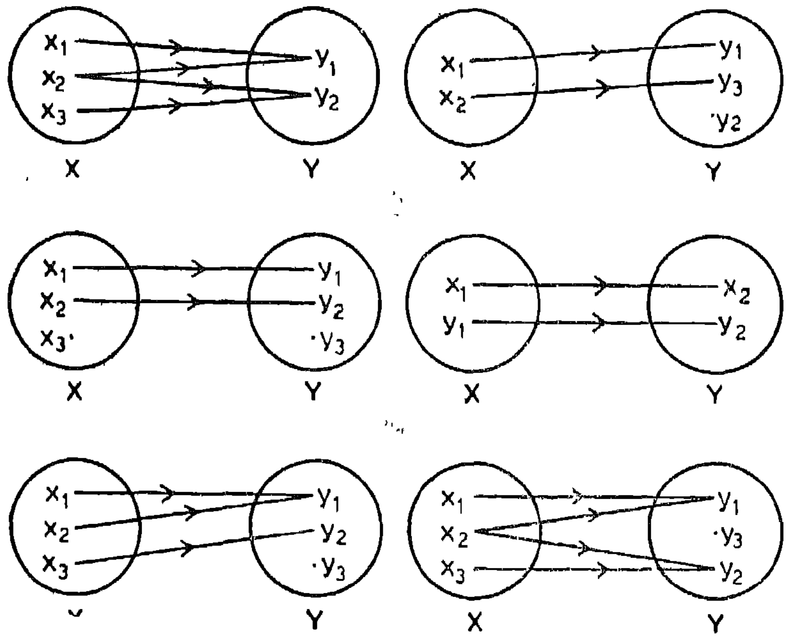


Fig. 6.2 to 6.7

is called a graph. Formally, the graph of a function is defined as follows:

*Definition*: The graph of a function f $x \to y$ is the set of points (x, f (x) ), in the cartesian plane, where x lies in the domain in which the function f is defined.

For example, the graph of the function $y = x^3$ is the set of points $(x, x^3)$ and the graph of the function $y = x$ is the set of points (x,x). In passing we may note that graph of the function y=x is a straight line.

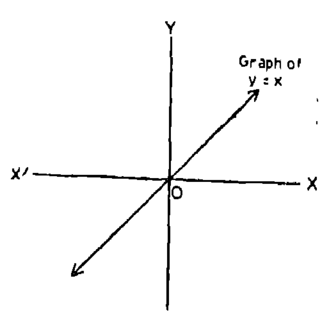



Fig. 6.8

Q. 1. In the diagrammatic representation of relations drawn in this section identify the examples and non-examples of a function. Support your answer.

Q. 2. Give four examples of relations (often found in an algebra textbook) which are not functions.

Q. 3. Why are $y = \pm \sqrt{x^2 + 2}$ and y= positive value of $\sqrt{-x^2-1}$ not functions?

Q. 4. To explain the concept of a function, it is better to consider the sets A and B other than the set of numbers in the beginning. Which of the following examples will you choose for teaching the concept of a function from the set A to the set B ?

(i) A= {x : x is a student of class VI}
B = { y y is the day of a week}
Y= f (x) means "x was born on the y-day"

(ii) B = {x : x is a woman}
A = {y. y is a man}
Y = f (x) means "y is the son of x"

(iii) A = { x: x is a student of Jawahar School}
B = { y: y is a teacher of Jawahar School}
Y = f (x) means "y is the teacher of x"

Q. 5. Give five more examples of a function from set A to set B, where A and B are sets other than the set of numbers

**Examples of Graphs of Functions**

Here we will illustrate the graphs of functions with the help of a few examples.

**Example 1** : The graph of the function

$$y=f(x) = |x|$$

Here the function can also be written as follows:

$$y=f(x) = \begin{cases} x \text{ if } x \geqslant 0 \\ x \text{ if } -x < 0 \end{cases}$$

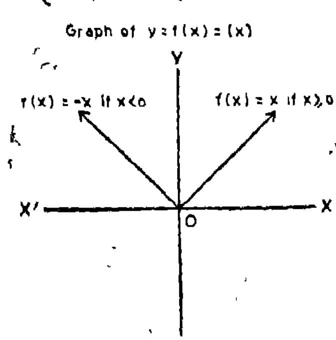



Fig. 6.9

Here it is of interest to note that at x=0 there is a *sudden* and *sharp* change in the direction of the graph. In higher mathematics, this property of the graph is the genesis of the concept of differentiability of a function.

**Example 2 :**

Graph of $y=f(x)=2x+3$

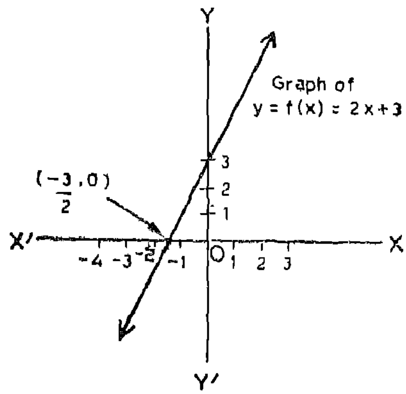



Fig. 6.10

Such graphs are called linear graphs as the graph is a straight line. Readers will remember that the graph of the function $y=f(x)=mx+c$ is a straight line.

**Example 3 :**

Graph of $y=f(x)=5$

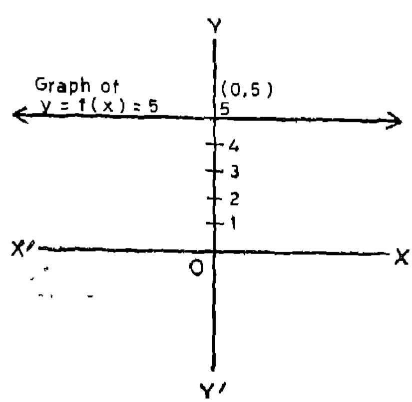



Fig. 6.11

This is a special type of linear graph. Here the value of f(x) is the same or constant for all values of x. So this is called the graph of a constant function. The general constant function is $y=f(x)=k$ (constant).

**Example 4 :** Graph $y=f(x)=x$

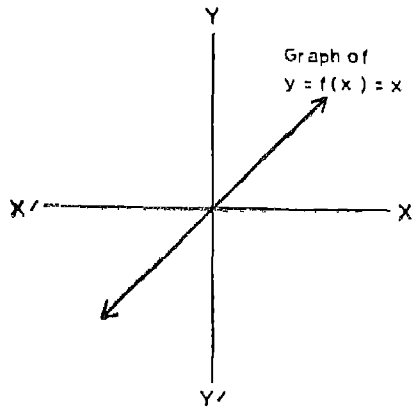



Fig. 6.12

In this graph at all points, the value of y equals the value of x. This is called the graph of identity function (which carries x to itself for all values of x).

**Example 5 :** Graph of $y=f(x)\leqslant(x)$ means the greatest integer less than or equal to x. For example.

$$(-1,2) = -2 \; : \; (-3) = -3$$
$$(2,5) = 2 \; : \; (4) = 4$$

It is to be noted here for any real number x such that $n \leqslant x < n+1$, $(x)=n$. So we can express this function as follows:

$$f(x)=n \text{ if } n \leqslant x < n+1, \text{ n being an integer.}$$

i.e. for $-2 \leqslant x < -1$, $x = -2$

for $-1 \leqslant x < 0$, $x = -1$

for $0 \leqslant x < 1$, $x = 0$

for $1 \leqslant x < 2$, $x = 1$

for $2 \leqslant x < 3$, $x = 2$

A hollow circle in the graph means that the point is not on the graph. A dot inside a circle means that the point *lies* on the graph.

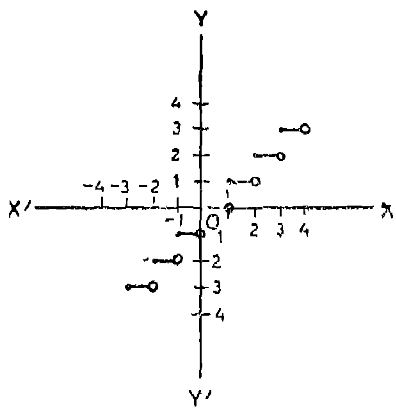



Fig. 6.13

This type of function is called *step function*, because the graph looks like a sequence of steps. (x) is called the greatest integer function. So the greatest integer function (x) is a step function.

**Example 6 :** Graph of

$$y=f(x)=\begin{cases} 1, \text{ if } x< 0 \\ 1.5, \text{ if } 0\leqslant x < 2 \\ 2, \text{ if } 2\leqslant x \end{cases}$$

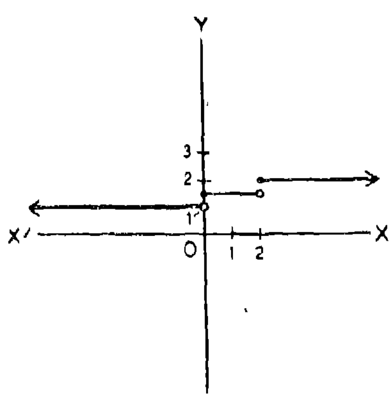



Fig. 6.14

Note that at the points x=0, and x=2, there are breaks in the graphs. Such points are called the points of discontinuity in the graph of the function and we say that the function f(x) is discontinuous at these points. Because of lack of continuity, these points are points of discontinuity. A rigorous definition of discontinuous function is beyond the scope of this book and may be found in any book of calculus.

**Example 7 :** Graph of the function

$$y=f(x)=\frac{1}{x} \quad (f . R\rightarrow R)$$

For example, let us determine the function corresponding to the following graph :

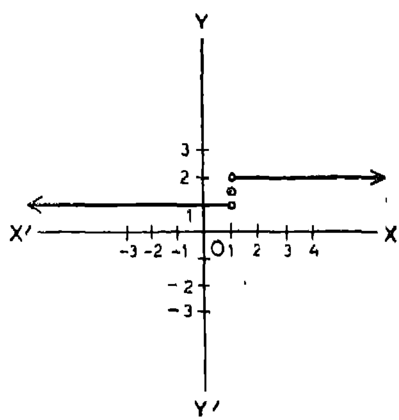



Fig. 6.15

If we remember the earlier examples carefully, we can write the function for this graph without difficulty as follows·

$$y=f(x)=\begin{cases} 1, \text{ if } x < 1 \\ 1.5, \text{ if } x = 1 \\ 2, \text{ if } x > 2 \end{cases}$$

---

Q. 1. The function f (x)=Sin x is defined as follows:

$$y=f(x)=\begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases}$$

Draw the graph of y=f(x) = Sin x and list the points of discontinuity.

Q. 2. Construct two step functions.

Q. 3. Given the domain, why is there only one identity function ?

Q. 4. Determine the function corresponding to the following graph :

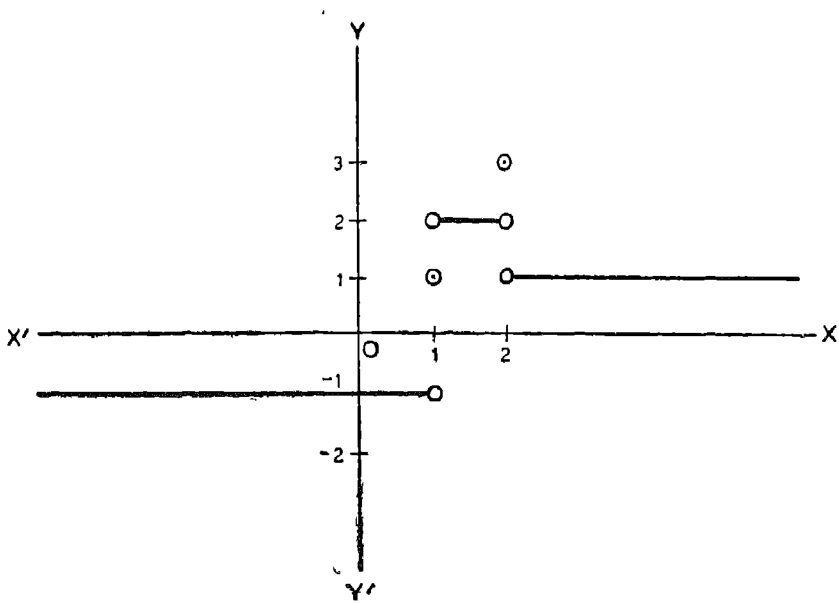



Fig. 6.16

---

Please note that the function $y=f(x) = 1/x$ is defined for all real values of x except $x=0$ The graph of this function is the set of all points whose coordinates are $(t, 1/t)$ where t is any real other than 0. The domain of this function is F the set $R - \{0\}$, R being the set of all reals. For positive values of x, f(x) is very large (or small) according as if x is very small

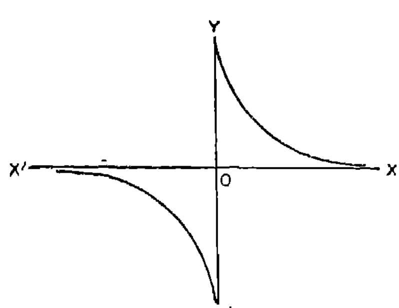



Fig. 6.17

(large). At $x=1$, $f(x) = 1$. Note that if $x < 0$, $f(x) < 0$. So the graph of the function will be as drawn in the figure 6 17.

The graph of this function is called a rectangular hyperbola. This graph sometimes represents in physics the relationship between pressure and volume of an ideal gas at a constant temperature.

This example is given here to impress that the graphs of functions need not be always straight lines— they can be curves as well. In fact very few of the graphs are straight lines.

*Note :* Though it is extremely difficult to determine the function corresponding to a given graph, theoretically every graph defines a function.

**Teaching Strategies**

Though in the development of the concepts we have not correlated the concepts with life situations, there is ample scope of correlating this lesson with daily life situations. For example, the temperature graphs used in hospitals are examples of graphs without breaks (continuous graphs i.e. graphs of continuous functions). Rainfall graphs or graphs correlating weight and height (where weight is approximated to the nearest kilograms and the weight increases with height) can be given as examples of step-functions If the class is a bright one, the notions of continuity and differentiability of a function can be correlated with the notion of graphs in a naive way. But this requires the teachers to be resourceful and imaginative.

Students sometime have a wrong idea that graphs should be drawn without a break. This wrong idea should be corrected by giving examples with breaks, e.g , step functions.

Care must be taken to impress upon the students that though due to pedagogic reasons almost all the examples given in this lesson are made of segments of straight lines, *very few* of the totality of graphs are composed of segments of straight lines In fact, many functions have curved graphs.

While teaching the definition of a function, teachers should take the help of diagrammatic representation of relations and non-examples It should be thoroughly explained under what conditions a relation ceases to be a function.

In the teaching of the definition, teachers are advised to start with examples and non-examples of a function and diagrammatic representation of relation and elicit the definition of function from the class in the development of the lesson Again this requires the ingenuity of the teacher.

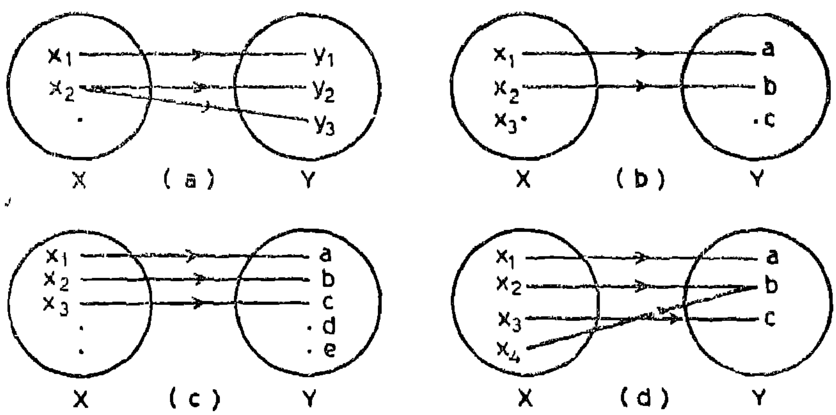
**Evaluation**

The important objectives of this lesson are to enable the students to:

(1) Define functions and graphs
(2) Give examples and non-examples of functions
(3) Draw the graph, given a function

**Specimen Test Items**

1. Which of the following are functions and which are not?




Figs. 6.18 to 6.21

2. Which of the following are f . R→ R?

(e) $y = f(x) = x^2+4$
(f) $y = f(x) = x+\sqrt{-2}$
(g) $y = f(x) = \log x$
(h) $y = f(x) = \sqrt{x-2}$

(Here R is the set of all reals)

3. Draw the graphs of the following functions:

(a) $y = f(x) = |x-2|$
(b) $y = f(x) = |x| - x$
(c) $y = f(x) = \begin{cases} x, \text{ if } x \text{ is even} \\ 0, \text{ if } x \text{ is odd} \end{cases}$

Here f(x) is a function f : I→I,
I being the set of all integers.

(d) $y = f(x) = \begin{cases} 2 \text{ if } x \leqslant 0 \\ 3 \text{ if } 0 < x \leqslant 2 \\ 5 \text{ if } 2 < x \end{cases}$

**Assignment for Teachers**

1. If $f(x) = \begin{cases} 0, \text{ if } x \text{ is rational} \\ 1, \text{ if } x \text{ is irrational,} \end{cases}$

   Can you draw the graph of this function as a curve without break (discontinuity)? Support your answer.
2. Why is the lesson important in the development of mathematics?
3. Write a lesson plan to introduce the definition of a function, starting with life situations and examples of relation.
4. How do non-examples play an important role in teaching functions and their graphs?
5. Mention five examples of functions used in Physics.
6. Mention five examples of functions which represent daily life situations.

## Quadratic Functions and their Graphs

**Introduction**

In the previous lesson we have discussed about functions and graphs. There we remarked that though most of the examples for graphs of functions in that lesson are composed of straight lines or segments of straight lines, most instances of graphs of functions are not so. These graphs are curves. One type of such functions whose graphs are curves and which are useful in discussion in physics is quadratic function In this lesson we will discuss what is a quadratic function and how the graph of a quadratic function looks like.

**Content Covered in this Lesson**

(1) Definition of a quadratic function
(2) Graphs of quadratic functions.

**Development of the Concepts**

*Definition of quadratic expressions :* Any algebraic expression of the form $a_0x^n+a_1 x^{n-1}+a_2\times x^{n-2}+ \ldots + a_{n-1} x+a_n$, where $a_0$, $a_1$———$a_n$ are all constants and x is a variable, is called a polynomial of n th degree in x.
*Definition .* An algebraic expression of the form $ax^2+bx+c$, where a,b,c are constants and x is a variable is called a quadratic expression.

Now it is quite clear that a quadratic expression is a polynomial of second degree in x. Sometimes the term 'quadratic polynomial' is used instead of 'quadiatic expression'.

---

Q.1. Give two examples of polynomials of (I) first degree and (ii) third degree

Q.2. Which of the following are quadratic expressions.

(i) $2x^3+3x$ (ii) $2x^4+5$

(iii) $2x^2$ (iv) 6

(v) $\frac{2}{x^2}+5$ (xi) $\frac{x^2+5}{x^2}$

(vii) $x^4+2x^2$ (viii) $x^2+4x+3$

---

**Graphs of Quadratic Functions**

Here we will discuss about the graphs of quadratic expression which is of the form $ax^2+bx+c$. In general, the graph of the function $y=f(x) = ax^2+bx+c$ is a parabola. It will be better if we take graphs of some particular cases of quadratic expression $ax^2+bx+c$.

**Example 1** . Graph of $y=x^2$

We draw up a table of values and plot these points Then we draw a smooth curve through them.

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| y | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

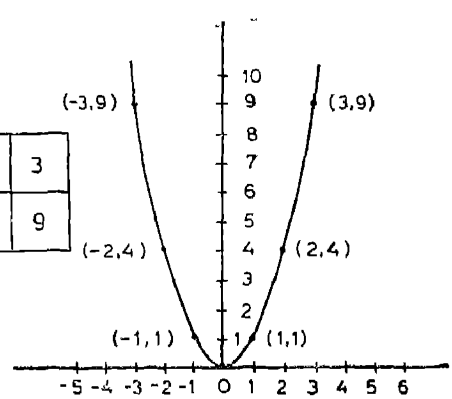



Fig. 6.22

The graph of $y=x^2$ is a parabola which is symmetric about the line $x=0$. The point O (o, o) is called the vertex of the parabola and the line $x=0$ is called the axis of the parabola. In this graph, corresponding to each value of x, the value of y is positive, and so the range of function is the set of all non-negative numbers.

---

Q.1. What is the rationale for drawing the table for points with integral coordinates only ?

Q 2. Plot seven points on the graph of the function $y=2x^2$

Q 3. Identify five points on the graph of the function for which the y—coordinates are greater than 100. Two of these five points should have negative x-coordinates.

---

**Example 2** · Graph of the function $y = ax^2$

*Case 1.* $a>0$.

(1) For any positive value of y, there are two values of x satisfying the equation which are equal in absolute magnitude, but opposite in sign. for $y=0$, $x=0$. For $y < 0$, there is no value of x satisfying the equation.

(2) For any given x, as 'a' increases, y increases.

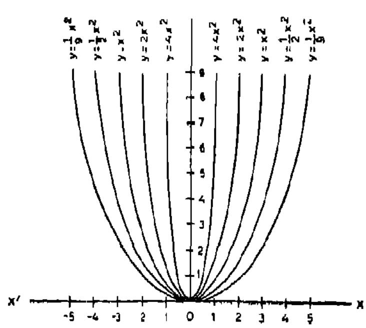



Fig. 6.23

Note that $(0, 0)$ is the vertex of all these curves and the line $x = 0$ is their axis.

*Case 2.* $a < 0$

(1) For each negative value of y, there are two values of x satisfying the equation which are equal in magnitude but opposite in sign, for $y=0$, $x=0$. For $y > 0$, there is no value of x satisfying the equation.

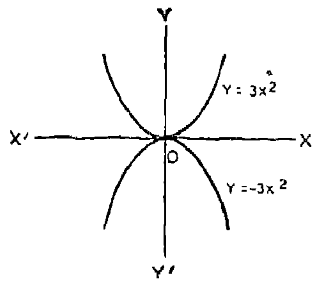



Fig. 6.24

Note that the graph of $y=-3x^2$ is the reflection of $y=3x^2$ about the x-axis. In general, (x,y) will be a point on the graph of $y = ax^2$ if and only if $(x-y)$ is a point on the graph of $y=-ax^2$.

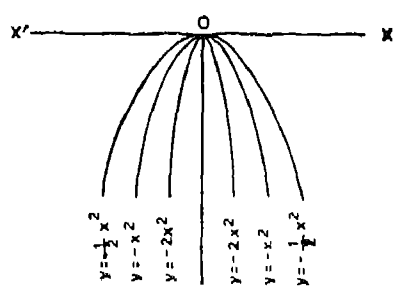



Fig. 6.25

Q.1. Plot the graph of the following :

(a) $y=2x^2$ (b) $y=-2x^2$ (c) $y=-\frac{1}{2}x^2$

Q.2. Determine a in the equation $y=ax^2$ so that the following points lie on the graph.

(a) (1, 1) (b) (1, 2) (c) (1, —3) (d) (2, 4)

**Example 3** : Graph of $y = ax^2+c$.

First draw the graph of $y=ax^2$. Then imagine that each point of the graph of $y = ax^2$ (and so the entire graph) is shifted parallel to the y-axis in the positive side by c units. The graph resulting due to this shifting will be the graph of $y=ax^2+c$.

Following figures will make the ideas involved in this shifting more clear.

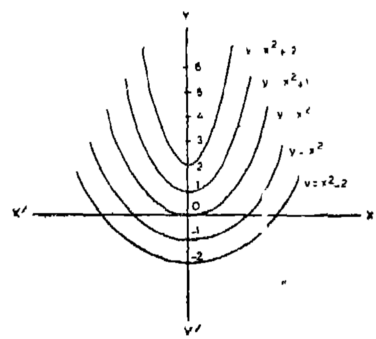



Fig. 6.26

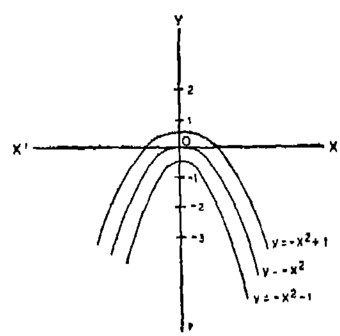



Fig. 6.27

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Q.1. Draw figures to show the relative positions of the graphs of functions in each of the following sets·

(a) $y=2x^2$, $\quad y=2x^2+1$, $\quad y=2x^2-3$

(b) $y=-2x^2$, $\quad y=-2x^2+2$, $\quad y=-2x^2-4$

---

**Example 4** · Graph of the function $y=a(x-k)^2$

Imagine that each point of the graph of $y=ax^2$ is shifted parallel to the x-axis in the negative direction (to the left) by k units. Then the resulting graph will be the graph of $y = a(x-k)^2$

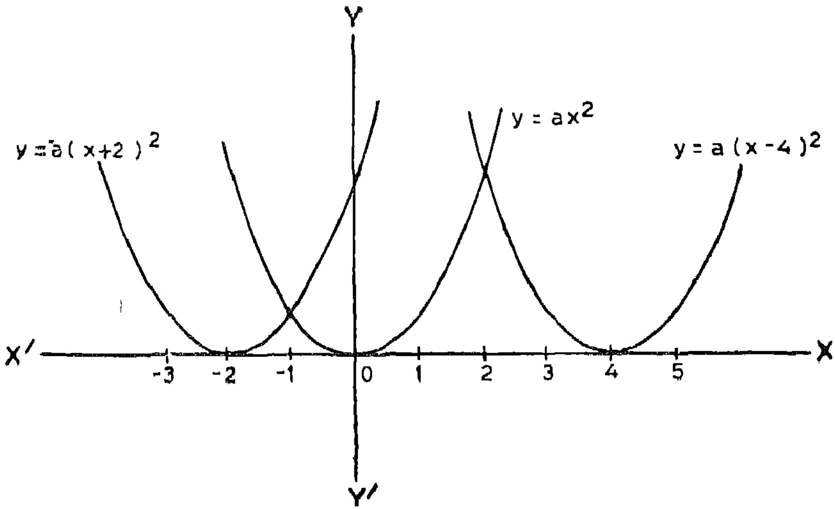


Fig. 6.28

The above figure makes the relative position of the graphs of the functions $y=ax^2$, $y=(x+2)^2$, $y=a(x-4)^2$ clear. Here $a > 0$

---

Q.1. Draw a figure showing the relative position of the graphs of the functions $y=-x^2$, $y=-(x+2)$, $y=-(x-2)^2$

---

**Example 5**. Graph of $y = a(x-k)^2+c$

Imagine that the graph of the $y=a(x-k)^2$ is shifted parallel to the

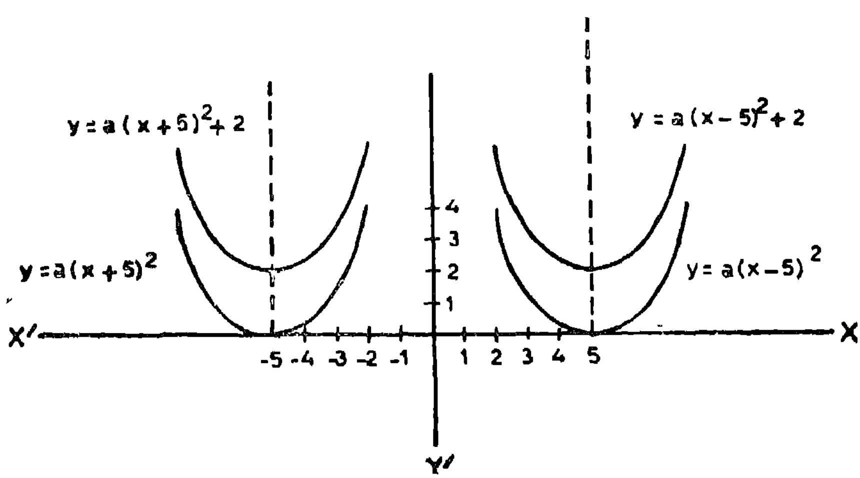


Fig. 6 29

y-axis in the positive direction by c units. The resulting graph will be that of $y=a(x-k)^2+c$.

The above figure explains the relative positions of the graphs of the functions $y=a(x-k)^2$ and $y=a(x-k)^2+c$. Note that in the above figure, $a>0$

---

Q. 1 Draw a figure to show the relative position of the graphs of the functions.

$$y=-2(x-1)^2, \qquad y=-2(x-1)^2+3, \qquad y=-2(x-1)^2-2.$$

---

**Graph of Quadratic Function**

The general quadratic function is denoted by $y=ax^2+bx+c$. It can be shown that if $a\neq 0$, then $ax^2+bx+c=a(x-k)^2+p$,

$$\text{where } k=\frac{-b}{2a} \qquad \text{and } p=\frac{4ac-b^2}{4a}$$

Once any quadratic function (expression) is put into the form $y=a(x-k)^2+p$, its graph can be drawn by the methods illustrated in the previous examples

---

Q. 1. Reduce the following functions in the form $y=a(x-k)^2+p$ and hence draw their graphs.

(a) $y=3x^2-6x+4$

(b) $y=-x^2+4x+6$

Q. 2. Prove the assertion that $ax^2+bx+c$ can be reduced to the form $a(x-k)^2+p$.

---

**Teaching Strategies**

In this lesson emphasis should be given to the skill in knowing about the nature of the graph by merely examining the quadratic function. Though for practice students may be asked to draw a few graphs on the graph paper, the treatment of the topic during the course of teaching needs to be theoretical in orientation.

Teachers should note that in this lesson examples given are in increasing order of complexity and examples are generalisations of previous examples. So the sequence of examples in order of teaching also suggests

that in teaching we move from particular to general For example $x^2$, $ax^2$, $a(x+k)^2$, $a(x+k)^2+p$ from a sequence in which each expression except the first one is a generalisation of the previous one. Teachers should remember that the generalisation of the concept B demands that A includes B and some more particular examples not covered by B.

While teaching the generalisations in this lesson, teachers should give the finer points about the relative positions of the graphs, whenever such occasion demands. These finer points can be drawn by examining a figure containing all the concerned graphs. This also helps the students to discriminate among the graphs.

Teacher should give enough practice in drawing graphs of quadratic functions. Again it is stressed that, even in case of lack of graph papers, drawing a graph means not necessarily actually drawing the graph, but analysing the graph and knowing its nature.

Teachers may correlate this lesson with the previous one on functions and graphs and emphasise the point that graphs of functions are generally curves rather than straight lines.

## Evaluation

The important instructional objective of this lesson is that students should be able (i) to draw the graphs of quadratic functions, and (ii) to discuss the nature of the graphs of quadratic functions.

## Specimen Test Items

1. Reduce the following to the form $a(x+k)^2+p$ and draw the graphs:

   (a) $y=2x^2$ (b) $y=-3x^2$

   (c) $y=2x^2+4x+5$ (d) $y=-4x^2-6x+7$

   (e) $y=7x^2+3x+12$ (f) $y=-2x^2-8x-3$

2. Compare and contrast the graphs in each of the following sets :

   (a) $y=x^2$, $y=2x^2$ $y=2x^2-3$

   (b) $y=3x^2$, $y=-3x^2+4$

   (c) $y=5x^2$, $y=5x^2-6$

   (d) $y=4x^2+3$, $y=4x^2+4x-3$

   (e) $y=4x^2+2$, $y=4x^2-2$

## Assignment for Teachers

1. Write a unit plan containing the subject-matter in this lesson and following the strategy that you will use for moving from one concept to another concept which is the generalisation of the previous one.

2 Students should be trained in discrimination when they are taught generalisation of concepts Explain this statement and illustrate by taking situations connected with this lesson.

## NUMBER SYSTEM

### Introduction

Primary students start studying mathematics with counting numbers and work out sums in arithmetic which involve four arithmetical operations (addition, subtraction, multiplication and division) with real numbers. Curiously enough, a rigorous discussion of these numbers forms a basis of higher mathematics and great mathematicians like Canchy, Cartor and Dedekind have contributed to the study of real numbers It is beyond the scope of this lesson to study the contributions of these mathematicians. Our approach in this lesson will be more functional. We will try to study the types of mistakes which may be committed by the secondary students in doing problems involving numbers, the reasons for these mistakes and how to remedy them. In short, we may say that our approach would be diagnostic in nature so that remedial instruction can be given to students in case of need. Our approach here becomes more meaningful as the number system is often included in secondary syllabus as a review chapter.

### Advantages of a Diagnostic Test

Before we actually come to the review study (which will be diagnostic) of the number system, it will be worthwhile to know the advantages of a diagnostic test

Diagnostic tests have two types of advantages (1) These tests help in finding the nature of errors committed by each individual student. Once the nature of errors of students are known, the teacher has to find the reasons for these errors. Then depending upon the reasons, the teachers can give remedial instruction to each individual student or to the class as a whole if the error committed is a common one, (2) The teacher can plan his teaching so that the students do not commit the errors. This is an important advantage which is preventive in nature Diagnosis of the errors committed by the students gives a guideline to the teacher in what respects he should modify teaching strategies.

Diagnostic tests for mathematics at the school level assume greater importance because students' deficiency in many school subjects can be traced back in many cases to deficiency in basic skills of reading and arithmetic "In mathematics, particularly, most of the deficiencies

may be traced to the weakness in basic facts and fundamental operations or understanding of algorithms". Also students may lack in good work-study habits and skills connected with mathematics.

**Diagnostic Items on Divisibility**

Obj. (1) To test the divisibility by 11.
Item (1) Which of the following number is divisible by 11 ?

(A) 11 11 111
(B) 345 345 654 654
(C) 1234 1234
(D) 666

Obj. (2) To test the divisibility by 4
Item (2) Which of the following is divisible by 4 ?

(A) 2348 34534
(B) 32238 8532
(C) 22 22 22 22 2
(D) 33 33 33 33 3

Obj. (3) To test the divisibility by 6
Item (3) Which of the following is divisible by 6 ?

(A) 234 834
(B) 344 133
(C) 31 48 22
(D) 332223

Obj. (4) To test the divisibility by 9
Item (4) Which of the following is divisible by 9

(A) 246342
(B) 322233
(C) 424323
(D)123434

Obj. (5) To test the divisibility by 5.
Item (5) Which of the following is divisible by 5 ?

(A) 1055553
(B) 2222222
(C) 3333333
(D) 4444445

**Diagnostic Items on Operations on Decimal Number**

Obj. (6) To locate a correct place of decimal in a product of two decimal numbers.

Item (6) Product of 001 and .00001 is

(A) .0001
(B) .000001
(C) .00000001
(D) .000000001

Obj. (7) To locate a correct place of decimal in a division, where divisor is 100.

Item (7) Express $\frac{32\ 4}{100}$ as a decimal number

Obj. (8) To locate a correct place of decimal in the product of two decimal numbers ending with 2 and 5.

Item (8) Product $0.02 \times .05$ is equal to

(A) .01
(B) .001
(C) .0001
(D) 100001

Obj. (9) To express a decimal number in the form $a \times 10^n$ where a and n are integers

Item (9) Express 3.24 in the form $a \times 10^n$ where a and n are integers

Obj (10) To express $(.01)^n$ in the form $10^k$ where k and n are integers.

Item (10) $(\ 01)^6$ is equal to

(A) $10^{-6}$
(B) $10^{-8}$
(C) $10^{-12}$
(D) $10^6$

**Diagnostic Items on Rationalisation of Irrational Numbers**

Obj. (11) To rationalise an irrational number of the form $\frac{a}{\sqrt{b}}$, where b is not a perfect square

Item (11) On rationalising $\frac{2}{5}$ , we get

(A) $\frac{2\sqrt{5}}{5}$ (B) $2.5^{-\frac{1}{2}}$ (C) $\frac{1}{\sqrt{(5/4)}}$ (D) $\sqrt{\frac{4}{5}}$

Obj. (12) To recognise an irrational number

Item (12) Which of the following is an irrational number ?

(A) $\sqrt{5}$ (B) $\sqrt{25}$ (C) $3\sqrt{125}$ (D) $4\sqrt{625}$

Obj (13) To simplify an irrational number in the lowest form.

Item (13) Select an irrational number out of the following

(A) $\frac{22}{7}$ (B) $\sqrt{10}$ (C) $\sqrt{121}$ (D) $\sqrt{625}$

Obj. (14) To rationalize $\frac{a}{\sqrt{b}-\sqrt{c}}$ where a, b and c are positive integers and b and c are not perfect squares.

Item (14) On simplification $3\sqrt{45}-\sqrt{20}+7\sqrt{5}$ we get

(A) $10\sqrt{30}$ (B) $14\sqrt{15}$ (C) $14\sqrt{5}$ (D) $8\sqrt{5}$

Item (15) On rationalizing $\frac{3}{\sqrt{5}-\sqrt{2}}$, it will be equal to

(A) $3(\sqrt{5}+\sqrt{2})$

(B) $\sqrt{5}+\sqrt{2}$

(C) $\frac{\sqrt{5}+\sqrt{2}}{3}$

(D) 1

**Diagnostic Items on HCF and LCM**

Obj (15) To find out the LCM of two numbers a and b, where a divides b.

Item (16) If a divides b, then LCM of a and b is

(A) a (B) b (C) a.b (D) a/b

Item (17) LCM of 3, 6, 9, 18 and 36 is

(A) 3 (B) 18 (C) 36 (D) 72

Obj. (16) To find out the HCF of two numbers a and b, where a divides b.

Item (18) If a divides b, then HCF of a and b is (A) a (B) b (C) a b (D) a/b

Obj. (17) To recognise the relationship between two numbers a and b and their HCF and LCM.

Item (19) If LCM of two numbers a and b is 'l' and their HCF is 'h', then the relation between a, b, l and h is

(A) $a \times b = l \times h$

(B) $a \times h = l \times b$

(C) $a \times l = b \times h$

(D) $a \times b \times h = l$

Obj (18) To compute HCF of two numbers a and b if their LCM is known.

Item (20) If LCM of '20' and 'b' is 100, their HCF will be
(A) 20 b (B) 100 b (C) b/5 (D) 5 b

**Teaching Strategies**

Since this lesson is a review of the old concepts which are basic pre-requisite for the other branches of school mathematics, diagnostic test items are very important in the teaching strategy.

Once the teacher comes to know the nature of and reasons for the errors of the students, he should take suitable remedial measures to remove the learning difficulties. After remedial teaching, he should give the students another parallel diagnostic test to measure the efficacy of remedial teaching. While constructing the diagnostic items, he should see that one item tests only one specific ability and is written in clear, unambiguous and simple language

The teacher should follow the three steps of the development of a diagnostic test. These steps are given below ·

(a) *Content Analysis* First step in the construction of a diagnostic test is content analysis. Content analysis of a unit requires the analysis of the unit into its sub-units or components.

For example, the subtraction of whole numbers may contain the following components.

(*i*) Concept and vocabulary of subtraction
(*ii*) Subtraction as the inverse process of addition
(*iii*) Subtraction not involving the use of carry-over principle
(*iv*) Subtraction involving the use of carry-over principle.

(b) *Writing items :* The next step is to write one or more test items to test the abilities in each component of the unit.

It should be remembered that each sub-test of a component should test the student's ability in that component and should be free from other components as far as possible There should be enough items in each sub-test to measure all the abilities related with a component. Items should be written in clear and simple language. If objective type items are to be given, great care should be taken in their construction.

(c) *Assembling the test* · When the items on various components of a unit are ready, they have to be assembled to form a test The items pertaining to any one component should be written together. As far as possible the components of the unit and also the items should be arranged

in order of complexity and difficulty level. In the test, guidelines and clear directions should be given regarding the purpose of the test, time allowed and the mode of answering. In order to make the scoring easy and objective one, a scoring key should be prepared for the test The scoring key may contain the correct answer (response) in the case of objective type item and a marking scheme for short answer essay type questions The scoring key may contain important points of the expected answers for the short answer and essay type questions. The test should be properly edited and reviewed (may be by the colleagues of the teacher) before it is sent for printing or before it is administered.

**Assignment for Teachers**

1. Prepare the content analysis and diagnostic test for the following topics in number system·
   (a) Computation with irrational numbers
   (b) Properties of whole numbers
2. Why should one diagnostic item test only one ability of the students? Illustrate with items on number system.

EQUATIONS AND INEQUATIONS

**Introduction**

We know that the children are using the symbols $>$ & $<$ in primary classes. They are also using 'The order idea' in division while finding the best multiple. The symbols of $>$ & $<$ are also used to form inequations like $ax+by<o$. The knowledge of order idea and method of solving linear equation in one and two variables are essential to study inequations. Inequations are used in the study of linear programming, which deals with the problems of maximisation and minimisation of the following types:

1 A farmer has 10 acres of land. He wants to sow mustard and wheat in his fields with an investment of Rs. 4,600/-. The expenditure on the mustard crop comes to Rs. 400/- per acre while on wheat crop it is Rs. 600/- per acre. He gets a profit of Rs 500/- per acre by the mustard crop, while Rs. 700/- by the wheat crop. In what areas he should sow the mustard and wheat crops?

2. Two tailors A & B earn Rs. 15.00 and Rs. 20.00 per day respectively. A can stitch 6 shirts and 4 pants per day, while B can stitch 4 shirts and 7 pants per day How many days shall each work if they want to produce at least 60 shirts and 72 pants at a minimum labour cost?

Such problems are called optimisation problems and are included in the linear programming We translate these problems into inequations, draw the graphs of these inequations and find the solutions.

We also know that the textbooks at elementary classes consist of the system of linear equations in two variables Students find unique solutions This is one type of linear equation in two variables. We will study in this unit the other types of linear equations in two variables.

**Content Covered in the Unit**

1. Sentence and Statement.
   (a) Meaning and types of sentences
   (b) Domain and solution set of an open statement.
   (c) Equations and Inequations.
2. System of Linear equations.
   (a) Linear equation in one and two variables.
   (b) System of linear equations.
3. Inequality
   (a) Order relation in the real number system
   (b) Inequality axioms and order properties of real numbers
   (c) Linear equation in one variable
   (d) Linear inequation in two variables.
4. Problems based on linear inequations
   (a) Region enclosed by three inequations.
   (b) Daily life problems.

**Development of the Concepts**

**1. Sentences and Statements**

(a) *Meaning and types of Sentences*: We build up sentences by using nouns, pronouns, verbs, adjectives etc. to express our thoughts. Likewise to express our thoughts in mathematics we use mathematical symbols, numerals and literal numbers—x, y, z, a, b, c. etc. There are two types of sentences—

(i) Sentences such as $3 + 4 = 7$, $3 + 8 = 11$, $4 > 9$, $3 < 8$ etc. which can be identified as true or false are called STATEMENTS.

(ii) Sentences such as $x + 3 = 4$, $x \geqslant 3$ etc which cannot be identified as true or false without introduction of additional information, are called **Open Sentences.**

The sentence $x \geqslant 3$ becomes $4 \geqslant 3$ for $x = 4$ which is true, the sentence $x \geqslant 3$ becomes $2 \geqslant 3$ for $x = 2$ which is false.

Symbol x can take values on a set of numbers and is called a **Variable** An open sentence contains a variable or variables.

Consider the sentence

$$3x + 12 = 3(x + 4)$$

Is this a sentence or a statement? If we know that x is a real number then the sentence is a true statement for all values of x. Such a sentence is called a **General Statement**

*Note I*. In some books the open sentence is also stated as Open Statement.

*Note II*, In some books $(3x + 12) = 3(x + 4)$ is placed in the category of open sentence because it consists of a variable x.

---

Q. 1. Monthly income of Ramesh is three times that of Hari.

(i) Translate it into mathematical sentence.

(ii) Is this an open sentence or a statement?

Q 2. Ramesh is a business man. His monthly income is not less than Rs. 2,000/- Translate this into mathematical sentence.

---

(b) *Domain of a variable and solution set of an open sentence* You have noted that an open sentence contains a variable. A variable that can be replaced by a real number is called a *real variable*. We say that the set of real numbers is the domain of a real variable. Similarly if a variable can be replaced by integers only, then the set of integers is the domain of the variable

*Note* · If the domain is not specified in a problem it is taken to be a set of real numbers.

The *solution set* of an open sentence is the set of those elements of the domain for which the open sentence becomes the true statement.

**Solution sets of an open sentence may be different for different domains.**

Solution set for $x + 2 = 5$ over the set of integers (3) If the domain is the set of negative integers, the solution set is [ ], i e, Null set. Thus for the same open sentence, the solution set may also be different for different domains.

---

Q. 1. Find the solution set for $x - 2 > 5$

(a) If the domain is a set of real numbers

(b) If the domain is a set of integers

Q 2. A man purchases x pens and y pencils A pen costs Rs. 2/- and a pencil Rs 0.50. To find the different possibilities of purchasing pens and pencils in Rs. 16/- We translate the problem into the open sentence $2x + \frac{y}{2} \leqslant 16$

(a) What is the domain of x ?

(b) What is the use of above problem in teaching domain of a variable ?

---

(c) *Equations and Inequations* · We use the symbols $=$, $>$, $<$, $<$ etc. in many open sentences. For example $x + 2 = 7$ is an open sentence of equality, while $x - 2 > 5$ is an open sentence of inequality.

*Definition* . An open sentence involving the relationship of only *equality* is called an *equation.*

*Definition* · An open sentence involving the relationship of order ($>$, $\not>$, $<$, $\not<$, $\geqslant$, $\leqslant$ ) is called an inequation

**2 Systems of Linear Equations**

(a) *Linear equation in one and two variables* An equation of the type $ax + by + c = 0$ where a and b are not simultaneously zero, is called a linear equation in two variables. For instance $y - 3x + 2 = 0$ and $2x + 5y = 0$ are linear equations in two variables

Is $lx + oy - 4 = 0$ a linear equation ?

If we compare this with $ax + by + c = 0$, we find a and b both are not zero simultaneously Here only $b = 0$ Hence it is a linear equation. The simplified form of the above equation is $x - 4 = 0$ This consists of only one variable. Thus we can say that it is a linear equation in one variable.

---

Q. 1. Express the following equations in the form $ax + by + c = 0$. where a,b,c, are real numbers.

(i) $3x - 5 = 0$

(ii) $5y + 3 = 0$

(iii) $x = 0$

(iv) $3y = 0$

---

(b) *System of Linear Equations* : Two linear equations of the form

$$a_1 x + b_1 y + c_1 = 0$$
$$a_2 x + b_2 y + c_2 = 0$$

Comprise a system of linear-equations.

If we draw the graph of a system of linear equations, we will get two straight lines There are three cases possible.

(i) Two lines can intersect at a point.

(ii) Two lines can be parallel.

(iii) Two lines can be overlapping

Can we decide the case by inspecting a system of linear equations ? Yes, we can. Now we will study how to do it

You can draw the graph of a linear equation in two variables. To draw the graph you can find two ordered pairs (coordinates of two points) that satisfy the given equation. We would then plot these points and join them to form a straight line. Look at the graph of

$$3x + 5y = 15$$

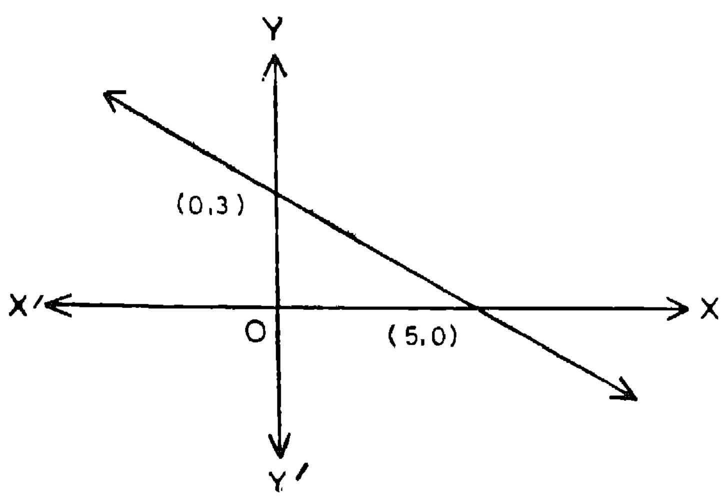


Fig. 6.30

The graph will be a straight line as in the figure. System of linear equations consists of two lines. Consider the following systems of linear equations and look at their graphs :

(1) $3x + 5y = 15$ (Fig. 6.31)

$6x + 10y = 30$

(2) $3x + 5y = 15$ (Fig. 6 32)
$6x + 10y = 19$

(1)

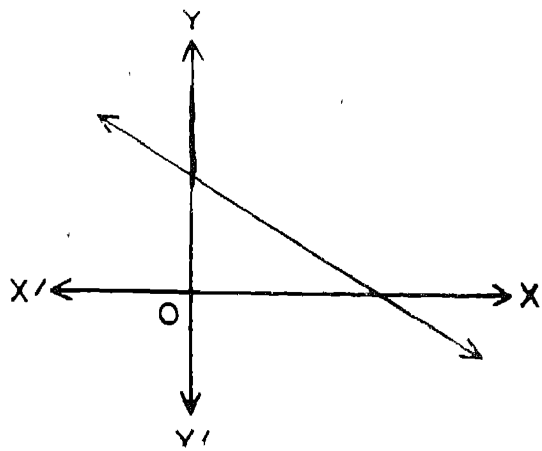


Fig 6.31

(2)

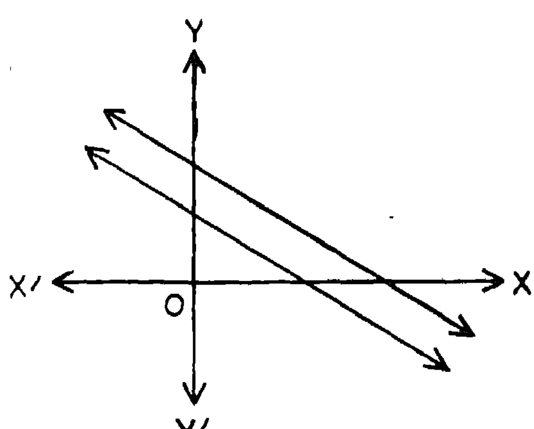


Fig. 6.32

(3) $3x + 5y = 15$

$6x + 2y = 9$ (Fig 6 33)

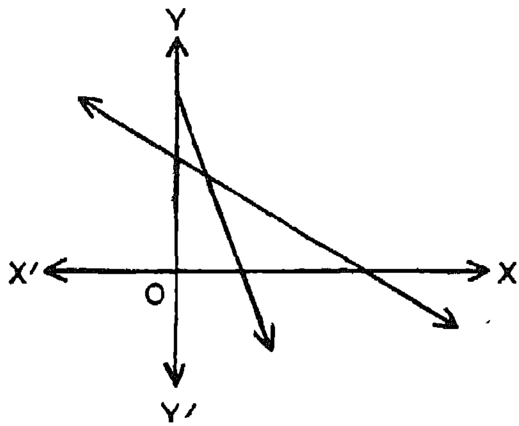


Fig 6.33

**Case (1)**

Every point on the graph of the first equation $3x + 5y = 15$ is also a point on the graph of the second equation.

There are, therefore, infinite number of points common to both the graphs. In other words, there are infinite number of solutions to the given system (1)

We observe that in the given system (1), if we multiply both the sides of the first equation by 2, we get the second equation. We can say that these linear equations are dependent.

A system of linear equations in which one equation can be obtained from the other by multiplying (or dividing) by an appropriate constant, is called a system of linear dependent equations. It has infinite solutions.

---

Q. 1 Select two linear equations out of the following to form a system of linear dependent equations.

$$2x - 4y = 3$$
$$x - 2y = 3$$
$$2x - 4y = -3$$
$$6x - 12y = 9$$

Q. 2. If $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ form a system of linear dependent equations and $a_1 = 3$ and $a_2 = 9$, then

1. Find the relation between $b_1$ and $b_2$

2. Find the value of $c_1$ if $c_2 = 3$
3. Is $a_1/a_2 = b_1/b_2 = c_1/c_2$ true for a system of linear dependent equations?

---

**Case (2)**. Look at Fig. 6 32. The lines are parallel. They have no common point and, therefore, the solution set of the given system is empty. This is called a system of linear inconsistent equations.

On comparing the coefficients of x and y of a system of linear dependent equations

$a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$

We find that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

On comparing the coefficient of x and y of a system of linear inconsistent equations, we find that

$$\frac{2}{4} = \frac{3}{6} \neq \frac{12}{9}$$

In general we can say that a system of linear equations $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ is inconsistent if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

---

Find the value of c, if

Q. 1. $3x + 5y = 8$ and $3x + 5y = c$ forms a system of linear inconsistent equations.

Q. 2. If $ax + 4y = 3$ and $3x + by = 4$ forms a system of linear inconistent equations, find the relation between a and b

---

**Case (3)**: In Fig. 6.33 lines meet at a point. In this case the solution is unique. You can verify if $a_1x + b_1y + c_1=0$ and $a_2x + b_2y + c_2=0$ has a unique solution then

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

---

Q. Is the system of linear equations

$$2x - 3y = 6$$
$$2y - 3x = 6$$

dependent, inconsistent or unique?

---

(3) **Inequality**

(a) *Order relation in the real number system :* We know that the order relation exists in the set of real numbers Suppose a and b are two real numbers, then $a > b$ if and only if $a - b$ is a positive real number If we assume 'd' a positive real number such that $a - b = d$, then $a = b + d$. You can illustrate this by taking some real numbers and marking points on a real number line.

---

Q 1. If $p > q$, then mark a point to represent 'q' on the following number line


Fig 6·34

Q. 2. $a \nless b$ is equivalent to (A) $a < b$, (B) $a = b$, (C) $a > b$, (D) $a \geqslant b$

Q 3. "A real number line is a tool to teach order relations". Discuss.

---

(b) *Inequality axioms and order properties of real numbers .* While solving linear equation in one variable, we use the following axioms ·

(1) If the same real number is added to both sides of an equation the resulting equation is equivalent to the original equation.

(2) If both sides of an equation are multiplied by the same non-zero real number, the resulting equation is equivalent to the original equation.

Similarly following propositions of real number are used in solving linear inequations in one variable.

1. If $a \geqslant 0$ and $b \geqslant 0$ then $a + b \geqslant 0$
2. If $a \geqslant 0$ and $b \leqslant 0$ then $a - b \geqslant 0$
3. If $a \leqslant 0$ and $b \leqslant 0$ then $a + b \leqslant 0$
4. If $a \leqslant 0$ and $b \geqslant 0$ then $a - b \leqslant 0$
5. If $a \geqslant 0$ and $b \geqslant 0$ then $a.b \geqslant 0$
6. If $a \geqslant 0$ and $b \leqslant 0$ then $a b \leqslant 0$
7. If $a \leqslant 0$ and $b \leqslant 0$ then $a.b \geqslant 0$

We also need the following properties of the real numbers to solve an inequation. These properties can be derived from the above propositions

($I_1$) If $a < b$ then $a + c < b + c$ for all real numbers a, b and c.

Proof By definition of $b > a$, there exists a positive real number 'd' such that $b = a + d$ On adding 'c' to both sides,

$$\left.\begin{aligned}(b + c) &= (a + d) + c\\ &= a + (d + c)\\ &= a + (c + d)\\ &= (a + c) + d\end{aligned}\right\} \text{properties of real numbers}$$

$\therefore \quad b + c > a + c$

or $\quad a + c < b + c$

Similarly, the following properties can be proved easily by using definition, propositions, and properties of real numbers.

($I_2$) If $a > b$ then $a + c > b + c$ for all real numbers a, b & c.

($I_3$) If $a > b$ and if $c > 0$ then $a.c < b.c$

($I_4$) If $a > b$ and if $c < 0$ then $a.c < b\,c$

($I_5$) If $a > b$ and if $c > 0$ then $a.c > b.c$

---

Q. 1. If $a < b$, then prove that $a + c < b + c$ for all real numbers a, b and c.

Q. 2. If $a < 0$ and $b > c$ then prove that $ab < ac$

Q. 3. Before proving the statement "If $a < 0$ and $b > c$, then $ab < ac$". It is useful to verify by taking particular values of real numbers a, b, c. Why?

---

(c) *Linear inequation in one variable* . An inequation involving one variable such that the variable appearing to the first power only is called a linear inequation in one variable. For example $2x - 3 < 7$ is a linear inequation in one variable.

What are the real values of x for which $2x - 3 = 7$ is a true statement ? We can get infinite values of x for which $2x - 3 = 7$ is true. If these are represented on a number line, the graph will be as below

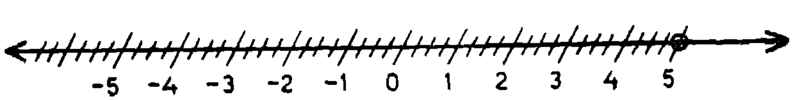


Fig. 6 35

Thus a solution set of $2x - 3 < 7$ over a real number is the set consisting of all the real numbers less than 5

**Hollow dot at 5 on the number line in Fig. 6.35 indicates that this is not the solution of the linear inequation.**

If the domain of x is a set of integers then the solution set of $2x—3 < 7$ will be $\{4,3,2,1, -1, -2, -3, ..\}$. The graph of the solution set will be as below

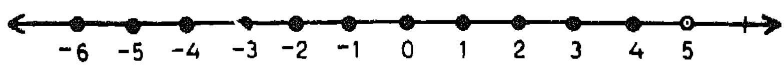


Fig. 6.36

To solve an inequation is to determine its solution set. The elements of the solution set are called the solutions of inequation.

To solve an inequation the propositions and the properties of real numbers relating to inequalities as stated above are used Now we will study the method of solving inequations.

**Example 1**

Solve $2x - 1 < 7$ and represent it on a number line

**Solution :** $2x - 1 < 7$

$$\rightleftharpoons 2x < 7 + 1 \text{ ( By } I_1)$$
$$\rightleftharpoons x < 8/2 \quad \text{( By } I_3)$$

Solution Set $= \{ x : x < 4 \}$

The solution set is illustrated on a real number line as below

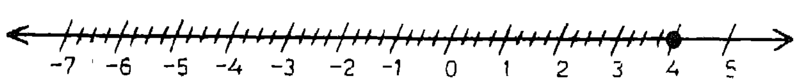


Fig. 6 37

**Example 2**

Solve : $2x—1 < 7$ and represent it on a number line for the set of positive integers as domain

**Solution :** $2x -1 < 7$

$$\rightarrow 2x < 7 + 1 \quad \text{(By } I_1)$$
$$\rightarrow 2x < 8$$

$\therefore$ Solution set $= \{ 1,2,3\}$

The solution is illustrated on a real number line as below

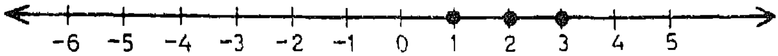


Fig. 6.38

**Example 3** .

Solve . $5x - 16 > -4x + 11$

Solution · If $5x - 16 > -4x + 11$

$\rightarrow (5x - 16) + 16 > (-4x + 11) + 16$ (By $I_1$)

$\rightarrow 5x > -4x + 27$

$\rightarrow 5x + 4x > (-4x + 27) + 4x$ (By $I_2$)

$\rightarrow 9x > 27$

$\rightarrow (9x)\ 1/9 > (27)\ 1/9$ (By $I_4$)

$\rightarrow x > 3$

$\therefore$ Solution set $= \{x \quad x > 3\}$

---

Q. Solve the following inequations and represent their solution on a number line

(i) $4x - 7 > 6x + 5$

(ii) $\frac{3x - 5}{2} > 6x - 8$

---

We have seen earlier that $x = 4$ is an equation of first degree in one variable and its graph is a point on a number line. Now consider the inequation

$$x \geqslant 4$$

The statement $x \geqslant 4$ is a composite statement consisting of two simple statements

$$x > 4 \quad \text{or} \quad x = 4$$

connected with the word 'or' Thus $x \geqslant 4$ means $x > 4$ or $x = 4$ or both, we call $x \geqslant 4$ an inequation. If x is an integer, the graph on the number line consists of isolated points, a few of which are shown below ·

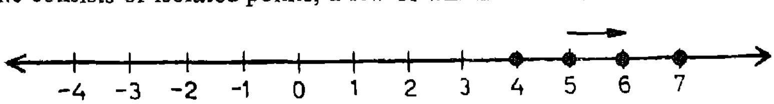


Fig. 6.39

If x is a real number it represents a ray as shown below .

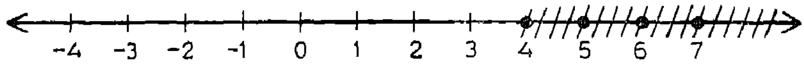


Fig 6.40

We can also plot the graph of the inequation $x \geqslant 4$ in the Cartesian plane. The graph of $x \geqslant 4$ will be the set of ordered pairs $\{ (x,y) : x \geqslant 4 \}$. Here again we may visualise two distinct situations

(i) where x, y are integers

and (ii) x,y are any real numbers.

**Case I** : Let x,y be integers.

The graph will consist of all those points of the plane for which $x \geqslant 4$, where x,y are integers. Some of the points which belong to the graph are marked with the point The arrows indicate that the graph consists of many other points—not shown in the figure

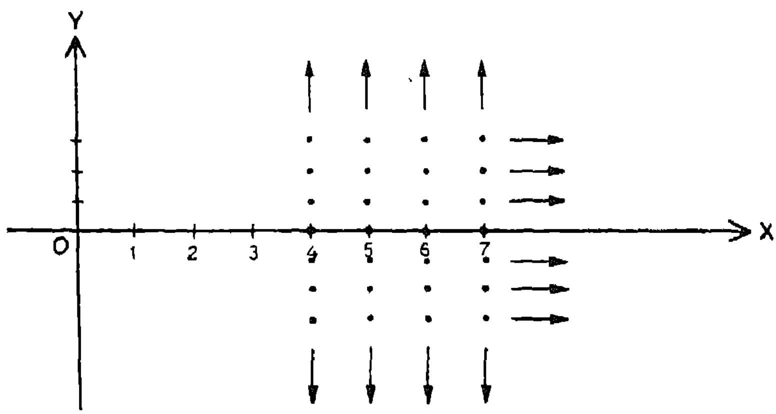


Fig 6.41

**Crse II** : Let x, y be real numbers.

Now the graph will consist of the region for which $x \geqslant 4$. The graph is shown by shaded portion in Fig. 6.42.

---

Q. Draw the graph of the following over Cartesian plane.

(a) $y \leqslant 4$

(b) $x > 0$

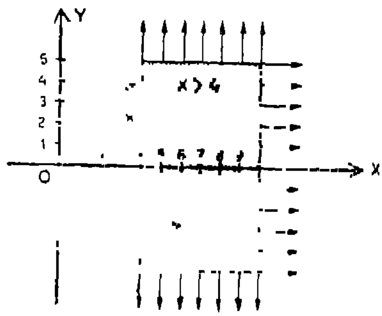


Fig. 6 42

(d) *Linear Inequations in two variables* · We know that $7x + 5y = 35$ is a linear equation in two variables x and y. If x and y are real numbers then the set of all ordered pairs of the form (x, y) which satisfy the equation $7x + 5y = 35$ will be on the line cutting the x-ax is and y-axis in Fig. 6 43.

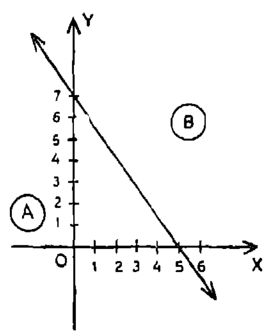


Fig. 6.43

Now we consider $7x + 5y > 35$ If we substitute the values of (x,y) for points on the cutting line none of them will satisfy the inequality $7x+5y> 35$. Can you find out all the points which will satisfy the inequation?

The line $7x + 5y = 35$ divides the whole plane into three parts ·

(1) Cutting line itself

(2) Region towards the origin (A)

(3) Region on the other side of the origin (B)

Let us consider a point not lying on the line say the origin (0, 0). On substituting the coordinates of the origin in the inequation, we find that $0 > 35$ which is false.

Can we say that all the points in the pair in which origin lies do not satisfy the inequation? Now we will discuss the question In general if x is the abscissa of P, then PL is

$$\frac{35 - 7x}{5}$$

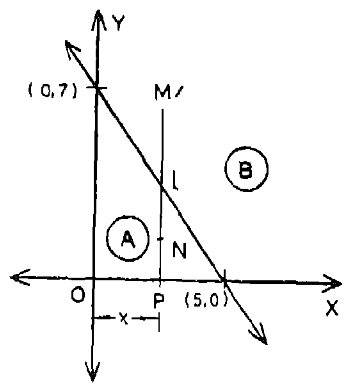


Fig. 6.44

If we take any point, N in the region A on the side of the origin, such that PN=$y'$ and so $y > y'$, then

$$PN < PL$$

$$y' < \frac{35 - 7x}{5}$$

or $5y' < 35 - 7x$

or $7x + 5y' < 35$

which is contrary to our inequation $7x + 5y > 35$. Therefore the point (x, y), in the region (A) will not satisfy $7x + 5y > 35$.

If we take 'M' in the region (B) in such a way that PM = $y''$ and so $y'' > y$, then

$$PM > PL$$

$$y'' > y$$

$$y'' > \frac{35 - 7x}{5}$$

or $5y'' > 35 - 7x$

or $7x + 5y'' > 35.$

This tells us that all the points (x, y") on the other side of the origin will satisfy the inequation. Thus the graph of the solution of $7x + 5y > 35$ will be as below ·

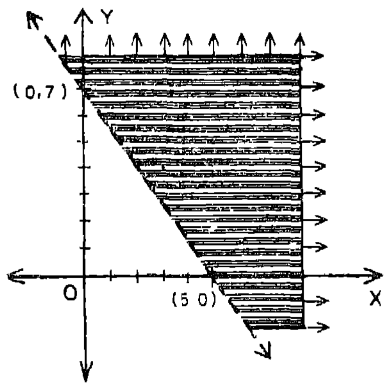


Fig. 6.45

What will be the graph of the solution of $7x + 5y \geqslant 35$ ?

In this case the coordinates (x, y) of the points on the line will also satisfy the equation along with the region for $7x + 5y \geqslant 35$. Thus the graph will be .

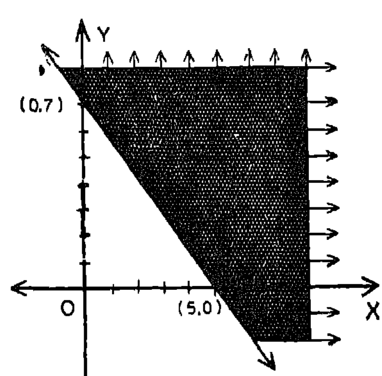


Fig. 6.46

What will be the graph of the solution of $7x + 5y < 35$ ?

Here the graph will consist of all the points of the region A. The graph will be as shown in Fig 6.47

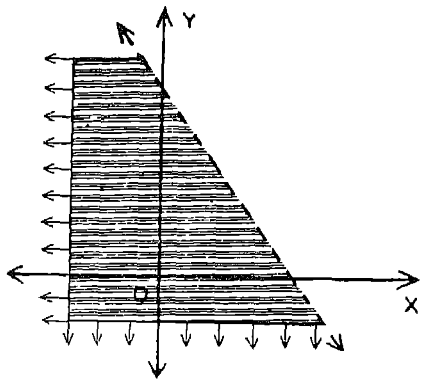


Fig. 6.47

Consider an inequation $3x \leqslant 2y$

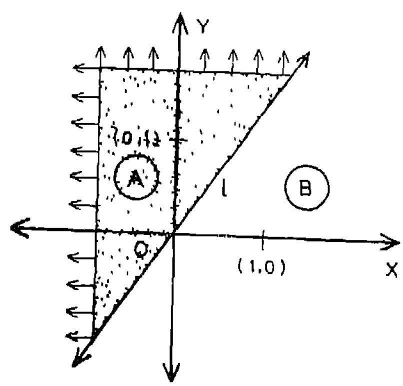


Fig. 6.48

Here also the line divides the plane into three parts l, (A) and (B) as in the above figure. Here we can observe that the point (0, 1) lies in (1) and the point (1, 0) lies in (B). On substituting (0, 1) in $3x < 2y$ we get

$$0 < 2$$

which is true. Thus region A will be the solution of $3x < 2y$.

---

Q. 1. Draw the graph of the following inequations .

(a) $x \leqslant 7y$

(b) $4x + 3y \geqslant 7$

Q. 2. While discussing about the region of the solution we have taken the general values y′ and y″. Can we derive the same result by taking the coordinates of two, three or more points ?

---

## 4. Problems Based on Linear Inequations of Two Variables

(a) *Region enclosed by three linear inequations :* We have seen that $x \geqslant 0$ represents a region lying towards the right of y-axis including the y-axis Similarly the region, represented by $y \geqslant 0$, lies above the x-axis and includes the x- axis. The question arises what region will be represented by $x \geqslant 0$ and $y \geqslant 0$ simultaneously.

Obviously, the region given by $x \geqslant 0$, $y \geqslant 0$ will consist of those points which are common to both $x \geqslant 0$ and $y \geqslant 0$. It is the first quadrant of the plane.

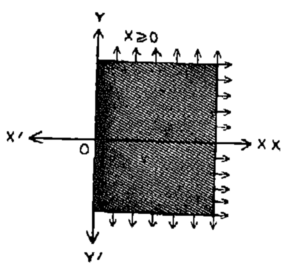


Fig 6.49

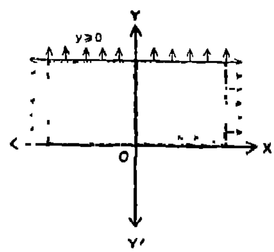


Fig 6.50

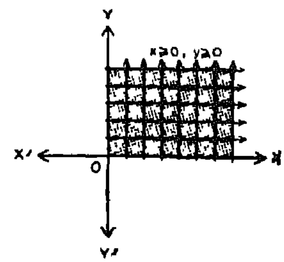


Fig. 6.51

Next we consider the region bounded by $x \geqslant 0$, $y \geqslant 0$ and $x + 2y \leqslant 8$

We have already seen that $x \geqslant 0$ and $y \geqslant 0$ represents the first quadrant. The graph given by $x + 2y \leqslant 8$ lies towards that side of the line $x + 2y = 8$ in which the origin is situated. Hence the shaded region given in the following figure represents $x \geqslant 0$, $y \geqslant 0$ and $x + 2y \leqslant 8$ simultaneously.

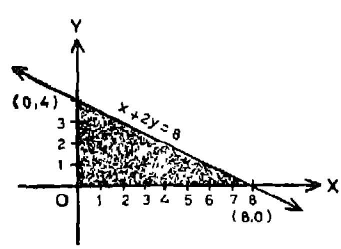


Fig. 6.52

If we consider the region bounded by $x \geqslant 0$, $y \geqslant 0$ and $x + 2y \geqslant 8$, then it will lie in the Ist quadrant and on that side of the line $x + 2y = 8$ in which the origin is not located. The graph is shown by the shaded region in Fig. 6.53.

**Problem** · Cost of a cup of coffee is 60 paise and cost of a cup of tea is 40 paise. What can we say about the number of cups of each type that can be taken for Rs 6 ?

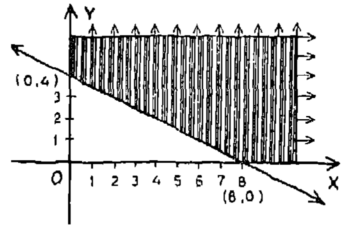


Fig. 6.53

**Solution** Let us assume that x cups of coffee and y cups of tea are bought.

$$x \geqslant 0 \quad (1)$$

and $$y \geqslant 0 \quad (2)$$

(1) No. of cups cannot be less than zero.

(2) No of cups will be in whole numbers.

The cost of tea and coffee in rupees would be

$$\frac{3x}{5} + \frac{2}{5} y$$

Since the total cost cannot exceed Rs 6

$$\frac{3}{5} x + \frac{2}{5} y \leqslant 6$$

or $$3x + 2y \leqslant 30 \quad (3)$$

We will examine the various possibilities for x and y by drawing the enclosed region between $x \geqslant 0$, $y \geqslant 0$ and $3x + 2y \leqslant 30$ in the Cartesian Plane.

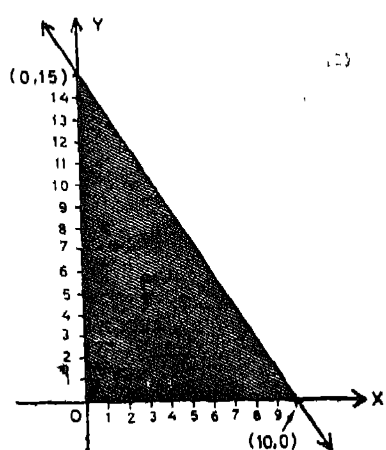


Fig. 6.54

From the graphs (Fig 6.54) it is obvious that the points in the first quadrant, lying in the shaded region and with integral coordinates are the number of cups of coffee and cups of tea that can be bought for Rs. 6.

1. What is represented by the point P ?
2. Mark the points for 13 cups of tea and 1 cup of coffee.
3. Mark the point for 10 cups of tea and maximum cups of coffee that can be taken for Rs. 6 from the graph.
4. Find the number of different possibilities for buying 8 cups of coffee and some cups of tea for Rs. 6 from the graph.
5. Find the total number of solutions from the graph.

**Teaching Strategies**

(A) Students have solved a system of these linear equations in class VIII in which the solution is unique. They must have observed :

(1) The graph of a linear equation is a straight line
(2) Two linear equations intersect at a point.
(3) The x and y coordinates of the point of intersection are unique for the given system of linear equations

While discussing a system of linear equation of other types, take any linear equation say $2x - 3y = 7$. Ask the students to draw the graph of the linear equation. Multiply the equation by a real constant to get another linear equation. Ask the students to draw the graph of the new linear equation. Similarly multiply by other constants and draw graphs in each case. Students will find that the graph of all these linear equations are the same straight line. You can now explain that a system of linear equations which represents the same straight line is called a system of linear dependent equations. Present a system of linear dependent equations and ask the students 'Can you say that it is a system of linear dependent equations' ? —only by inspection. Since the students have obtained a system of linear dependent equations only on multiplying by the constant, it is expected that they will give the correct answer.

Now present the system of linear equations of the form—

$$ax + by + c = 0$$
$$dax + dby + k = 0 \text{ where } k \neq dc$$

Ask them—'Is this a system of linear dependent equations?' Let the students draw the graph of the above type of linear equations. For example, consider a system of linear equations :

$$2x + 5y + 4 = 0$$
$$4x + 10y + 3 = 0$$

The graph will be two parallel lines. Ask the students to find a common point of these lines It will not be possible for them to find a common point and hence introduce such a system as a "system of linear inconsistent equations". Give enough exercises to find out a system of linear dependent equations and a system of linear inconsistent equations only by inspection. This is inductive approach.

Let us list the characteristics of inductive approach. We present a set of examples and the student reaches a conclusion by observing these examples. The conclusion will be in the form of a rule.

---

Q 1 Due to oversight some students say that the following system is a linear dependent system.

$2x - 3y = 5$

$6y - 9x = 15$

Can you think of some way to rectify such errors?

Q. 2 Write a test-item to test the following objectives·

The student can discriminate between a system of linear dependent and linear consistent equations.

Q. 3. Write the activity performed by a teacher in teaching a lesson by inductive approach.

---

(B) There are two methods of solving an inequation—

(1) Direct proof, and

(2) Graphical

Before solving an inequation it must be transformed into a standard form, by using commutative, associative and distributive laws of addition and multiplication of real numbers at the first instance.

| Type of Inequations | Standard Form |
|---|---|
| Linear inequations in one variable | $ax + b > 0$ |
| | $ax + b \geqslant 0$ |
| | $ax + b < 0$ |
| | $ax + b \leqslant 0$ |
| Linear inequations in two variables | $ax + by + c > 0$ |
| | $ax + by + c \geqslant 0$ |
| | $ax + by + c < 0$ |
| | $ax + by + c \leqslant 0$ |

**Direct Proof**

We have the set of axioms and theorems related to the relationships of less than and greater than of real numbers. (See pages 254-255) "These are used in solving inequations and this method is called Direct proof or proof by implication"

While proving a statement you write a sequence of statements, each statement being justified by an axiom or a previous theorem The proof ends with the statement of conclusion.

(See examples 1,2,3 on pages 256-257)

We should provide enough opportunity to our students for using the axioms and properties.

**Solving Inequations by Graphical Method**

1. For solving linear inequations in one variable, the graph of $ax + b = 0$ is drawn. This is a single point on real number line. After marking the point $x = -b/a$, it is to be decided

 (i) whether the points on the left or on the right of $x = -b/a$ will be included in the solution set.

 (ii) whether the point $x = -b/a$ will be included or will not be included in the solution set.

This may be decided by taking some points and substituting

2. For solving linear inequations in two variables the graph of the equation $ax + by + c = 0$ is drawn. After drawing the line $ax + by + c = 0$, it is to be decided:

 (i) whether the region containing the origin or the other region (B) will be included in the solution set

 (ii) whether the points on the line $ax + by + c = 0$ will be included or will not be included in the solution set.

To determine the region in which a given inequation is satisfied, emphasize the following procedure:

(1) Select an arbitrary point in one of the regions.

(2) Substitute the coordinates in the given equation.

(3) (a) If the given inequation is satisfied, the region selected is the desired region.

 (b) If the given inequation is not satisfied, the other region is the desired region

**Evaluation**

By learning this unit we expect many behavioural changes among students. Some of the behavioural changes are given below :

1. Students can write a solution set of open sentence for a given domain.
2. Students can identify the dependent and inconsistent systems of linear equations
3. Students can shade the solution sets of linear equations in a Cartesian plane.
4. Students can prove an inequality by using an axiom and properties of real numbers.
5. Students can translate optimization problems into inequations.

**Specimen Test Items for the Unit**

1 Solve $\frac{2}{3}x - 5 < 7 - 2x$

where x is a whole number. Also draw the graph.

2. Prove that —
   if $c < 0$ and $a > b$ then $a - c > b - c$
3. Shade the region on a real number line for which both of the following inequations are true :

   $x - 2 > 0$ and $x + 3 \geqslant 0$

4. Match the system of inequations with their type.

| | |
|---|---|
| (A) $3x + 7y = 15$<br>$2x + 5y = 3$ | (a) Equations having unique solution. |
| (B) $5x - y = 3$<br>$15x - 3y = 9$ | (b) Linear inconsistent equations. |
| (C) $2x + 3y = 8$<br>$8y + 12x = 32$ | (c) Linear dependent equations. |
| (D) $2x + y = 8$<br>$4x + 2y = 7$ | |
| (E) $x + y = 7$<br>$x + y = 12$ | |
| (F) $x = 0$<br>$y = 0$ | |

5. If $ax + by = 4$ and $3x - 2y = 8$ are linear dependent equations, find the value of a and b.
6. Explain the difference in the graphs of

$$ax + by > c$$

and $ax + by < c$

without drawing the graph.
7. By using axioms and properties of inequations prove that "if $a > b$ and $c < 0$ then $ac < bc$"
8. Convert $\frac{3}{2}x - 5 > 3 + x$ into standard form and solve it

Illustrate the solution on a real number line.

**Assignment for the Pupil-teacher**

1. To draw the graph of a line $2x - 7y = 24$, we need only two points to satisfy the equation Why?
2. Shade the common region of the following inequations

$$x \geqslant 0$$
$$y \geqslant 1$$
$$y \geqslant x + 1$$

3. Formulate all the inequations to solve the following problem —

"The floor of a hall measures 10 m by 10 m. A man wants to put in two types of cots in the hall — one occupying floor area of 2 m by 1 m and the other 2 m by $1\frac{1}{2}$ m. What are the different possibilities for putting the cots in the hall?"

4. Write an inequation for the following shaded region —

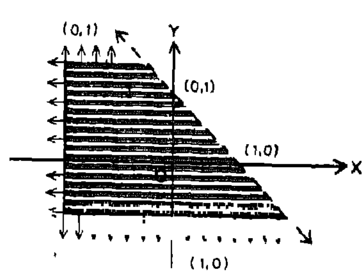


Fig. 6.55

5. Is there any similarity between inductive approach of teaching and Mathematical Induction?

## Algebraic Structures

### Introduction

In school we study different types of number systems— the system of natural numbers, the system of integers, the rational number system and the real number system These number systems possess certain properties with respect to the operations of addition and multiplication (on them) Without these properties algebraic computation in these systems would not be possible. The whole source of algebraic computation is based on these properties.

We can, if we like, seek to abstract what is algebraically essential and common to the number systems specified above and develops algebraic results which hold for each of these systems and study them without repeating the work for each particular system. This approach of abstraction is very important in modern mathematics and is characteristically typical of modern algebra which is sometimes, also called abstract algebra. The systems which we consider for our generalised or abstract study of this sort are called algebraic systems or algebraic structures. In this lesson, we will discuss the notions of two very important algebraic structures—group and field

### Content Covered in this Lesson

(1) Definition of a group
(2) Examples of a group
(3) Definition of a field
(4) Examples of a field.
(5) Solutions of simple equations in one variable in the field of real numbers.
(6) The notion of algebraic structures.

*Note* : The content of this lesson is developed on the assumption that the readers are acquainted with the different types of number systems. So readers are advised to refresh their knowledge of number systems before reading this lesson.

### Development of the Content .

**Group** . Let us consider I, the set of integers. If we add or multiply two integers, we always get an integer as the result of adding or multiplying. But if we divide 3 by 5, the result is $\frac{3}{5}$, which is not an integer. We exprsess this fact by saying that "I is closed under addition and multiplication, but not closed under division".

More generally, an operation on a set S is a rule which associates every ordered pair (a,b) (where $a \in S$, $b \in S$) to an element c of S. Symbolically, we say that $a * b = c$. It should be noted here that it has been assumed in this definition of an operation that S is closed under the operation*. So under this definition of an operation, addition and multiplications are operations on I, whereas division is not.

---

Q 1 Which of the following sets have addition, subtraction, multiplication and division as operations on them

(a) The set of natural numbers.

(b) $\{1, 0, 2, 3\}$

(c) The set of rational numbers.

(d) $\{1, 0, -1\}$

(e) $\{-1, -2, 0, 1, 2\}$

(f) $\{1, 0, i, -i, -1\}$

---

**The Notion of Subtraction as the Inverse Operation of Addition**

When you write $3-2=1$, you say 2 is subtracted from 3. Let us try to understand this a little better. 3 and 2 are positive integers $-2$ is the negative of the positive integer 2. Easily it is seen that $3 + (-2) = 1$.

Thus $3-2 = 1 = 3 + (-2)$

So when 2 is subtracted from 3, what we are actually doing is —we are adding 3 and the negative of 2. In general, if a and b are any two numbers, subtracting b from a means adding a and the negative of b. Symbolically, we can write :

$$a-b = a + (-b)$$

In a similar fashion we can argue that dividing the number a by the number b means multiplying the number a by the reciprocal $\frac{1}{b}$ of the number b. Symbolically, we can write :

$$a \div b = a \times \frac{1}{b}$$

---

Explain in detail, how

$$a \div b = a \times \frac{1}{b}$$

---

### The Inverse Element and the Identity Element

Consider the set I of integers under the operation of addition. We notice that $0 + a = a + 0$ and $a + (-a) = a - a = 0$ for every integer 'a' Here we say that (i) a is the additive inverse of a, i.e , the inverse of a under the operation of addition. (ii) 0 is the additive identity of I, i.e., the identity element of I under addition.

Similarly, consider the set $Q^+$, of non-zero positive rationals under multiplication We have

$$a \times 1 = a = 1 \times a$$

$$a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$$

for any and every non-zero positive rational $a \in Q^+$. Here we say that (i) $\frac{1}{a}$ is the multiplicative inverse (i e inverse under multiplication) of a, and (ii) 1 is the identity element of $Q^+$ under multiplication.

In general, let us suppose that * is an operation defined on a set S. Then if for every $a \in S$, $ae = a = ea$ where $e \in S$, we say that e is an identity element of S under the operation* Also if for an element $a \in S$, there exists an element $b \in S$ so that $a * b = e = b * a$, we say that b is the inverse element of a and vice versa. We briefly denote 'inverse element' and 'identity element' as 'inverse' and 'identity' respectively.

---

Q. 1. Define the inverse element and identity element with respect to an operation.

Q. 2. Determine the identity elements (if any) and inverse elements (if any) of the elements of the following sets under multiplication ·

(i) $\{0, 1, -1\}$
(ii) $\{1, -1\}$

---

### The Definition of a Group

In any set of numbers, we see that $(a+b)+c = a+(b+c)$. For example $(2+3)+5 = 2+(3+5)$. What is to be emphasised here is that given three numbers a,b,c and $d=a+b$, $e = b+c$, we have $d+c = a + e$. This property of numbers is called the associative property of numbers.

In general, if in a set S with respect to the operation*, we have

$$(a * b) * c = a * (b * c)$$

Whenever $a,b,c \in S$, the operation * is said to be associative in S.

Now we come to the definition of a group. A set S under the operation is said to be a *group* if

(i) $*$ is associative in S.

(ii) There exists an identity element in S with respect to $*$.

(iii) For every element $a \in S$ there exists an element $b \in S$ which is the inverse of a under $*$. b is generally denoted by $a^{-1}$.

Now we will give some illustrations.

**Example 1**: The set of all rationals under addition is a group.

**Example 2.** The set of all rationals under multiplication is not a group.

**Example 3**: The set $\{0, 1, -1\}$ is a group under multiplication.

**Example 4**: The set $\{0, 1, -1\}$ is not a group under addition, because addition is not an operation on the set

---

Q 1. Which of the following sets are groups?

(a) The set of all reals under multiplication.

(b) The set of all positive integers under multiplication

(c) The set 1, $-1$, i, $-i$, under multiplication

(d) The set 1, $-1$, i, $-i$, under addition

(e) The set of all three dimensional vectors under vector addition.

(f) The set of all three dimensional vectors where $\vec{a} * \vec{b} = \vec{a} \times \vec{b}$ (cross-product of $\vec{a}$ and $\vec{b}$).

Q. 2. Why do we, in the development of the concept of a group, start with the preliminary concepts like operation, inverse, identity and associative property and then come to the definition of a group? Why have we not started with the definition of a group?

---

**Cayley's Table**

Now a question arises, how we can describe the data which define a given group Cayley answered this question when he introduced the multiplication table of a group. This is called Cayley's table and is very much similar to the usual multiplication table. The elements of the group are written in the top row and, in the same order, in the left column of the table, and the entries in the table represent the products of the elements under the operation of the group

Consider the group 0, 1 under the operation of the ordinary multiplication of numbers.

Its Cayley's table will be as shown in Fig. 6.56

Similarly, the Cayley's table for the group 0, 1,—1 under multiplication will be as given below in Fig. 6.57.

2nd FACTOR

| | X | 0 | 1 |
|---|---|---|---|
| 1st FACTOR | 0 | 0 | 0 |
| | 1 | 0 | 1 |

Fig. 6.56

2nd FACTOR

| | X | 0 | 1 | -1 |
|---|---|---|---|---|
| 1st FACTOR | 0 | 0 | 0 | 0 |
| | 1 | 0 | 1 | -1 |
| | -1 | 0 | -1 | 1 |

Fig. 6.57

Cayley's table is useful in verifying if for all the elements a, b of a group S, a * b = b * a i.e., whether any two elements a and b of S commute.

*Definition* : If in any group S, a * b = b * a whenever a ε S, b ε S, then S is said to be a commutative or abelian group.

Cayley's table for any commutative group is such that entries of the table are located symmetrically with regard to the diagonal which runs from the upper left corner to the lower right corner. For illustration, we can take the Cayley's tables discussed.

2nd FACTOR

| | X | 0 | 1 |
|---|---|---|---|
| 1st FACTOR | 0 | 0 | 0 |
| | 1 | 0 | 1 |

Fig. 6.58

2nd FACTOR

| | X | 0 | 1 | -1 |
|---|---|---|---|---|
| 1st FACTOR | 0 | 0 | 0 | 0 |
| | 1 | 0 | 1 | -1 |
| | -1 | 0 | -1 | 1 |

Fig. 6.59

You may question or ask why we should have a discussion about commutative group, because any group of numbers under ordinary

addition or multiplication is necessarily commutative. But it must be noted that though it is difficult to give a commutative group with finite number of elements, it can be verified that the set of all non-zero matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a,b,c,d are integers forms a group under matrix multiplication and this group is not commutative as matrix multiplication is not necessarily commutative

---

Q. 1 Prepare Cayley's tables for the following groups·
(a) $\{1, i, -i\}$ under multiplication
(b) $\{1, w, w^2\}$ under multiplication
where $w^3 = 1$.

Q. 2. If you try to form a Cayley's table for a set A which is not a group under an operation, what special properties this table can have which indicate that the set A is not a group?

Q. 3 Verify by actual computation that the set of all matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers does not form a commutative group under matrix multiplication.

---

**Field**

Up to now in this lesson we have considered examples of systems denoted by sets with only one operation. Here we will consider sets over which two operations have been defined.

If we consider the set R of all real numbers with the usual addition and multiplication as operations on it, we have the following rules connecting the two operations viz. addition and multiplication

$a \times (b+c) = a \times b + a \times c$

$(b + c) \times a = b \times a + c \times a$

Whenever $a,b,c \in R$, the set of all reals, these rules of real numbers are called the left and right distributive laws. If the left and right distributive laws hold, we also express the same by saying that multiplication is distributive over addition. These laws play an important role in computational algebra as they are the connecting link between the two otherwise seemingly unrelated operations.

Now we can verify that for the set R of all real numbers or for the set Q of all rationals, the following conditions hold·

(i) Both R and Q are commutative groups under addition.

(ii) If we exclude O from R and Q, the resulting sets are called R—{O} and Q—{O}. Both R—{O} and Q—{O} are commutative groups under multiplication.

(iii) Left and right distributive laws hold in R and Q both

We can also check that the R†, the set of all positive reals and Q†, the set of all positive rationals do not satisfy all the above three conditions. We say in brief that both R and Q are fields under usual addition and multiplication, whereas R† and Q† are not.

Now we come to the definition of a field, which is a very important algebraic structure with two operations, usually called addition and multiplication and denoted by (+) and (·) respectively. You should be careful to note that while defining a field we take the symbols (+) and (·) (addition and multiplication) as **any two** operations on **the** same set, and **not necessarily the usual adddition and multiplication.**

The formal definition of the algebraic structure called 'field' is given below :

A non-empty set F with two operations defined on it which are usually called addition and multiplication and denoted by (+) and (.) respectively is called a '**field**' if the following conditions are satisfied

(i) F is a commutative group under addition.

(ii) F—{Z}, where Z is identity of F under addition, is a commuative group under multiplication.

(iii) Left and right distributive laws hold alternatively, i.e., "Multiplication is distributive over addition".

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Q 1. Which of the following sets are field ?
Support your answer.
(i) The set of all integers
(ii) The set of all rationals.
(iii) The set of all positive reals
(iv) The set of all positive rationals.

Q. 2. Why have we given non-examples of a field in the development of this concept ?

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**Algebraic Structures and Identities and Equations**

Now we know about two types of algebraic structures called group and field. These are examples of a "**composite object**" of a non-empty set A and one or two operations in A. Precisely the term "Composite object" is to be taken here to mean either an ordered pair of the form

$\{A, +\}$ in the case of a group or an ordered triplet $\{A, +, .\}$ in the case of a field.

*Definition* Such a composite object is called an algebraic structure with one operation or two operations (as the case may be).

Groups and fields are not the only types of algebraic structures. By changing the conditions to be satisfied by the set A under operation (s) we can form other algebraic structures. The examples of such algebraic structures are ring and integral domain which are not discussed here.

It is not without any reason that we study algebraic structures. The study of these structures throws light on the structure or system in which we work while doing mathematics. As an illustration, let us take solving linear equations in one variable.

While doing computation with equations or equalities we use the following properties of equality (=) in the field of all reals though we are sometimes unaware that we are working with respect to a field of all reals

We define equality on the field R of real numbers as given below

For $a,b \in R$, $a=b$ if and only if $a-b=0$

The relation = possesses the following properties .

1. For each $a \in R$, $a=a$ (Reflexivity)
2. If $a=b$, then $b=a$ for any a, b, $\in$ R. (Symmetry)
3. If $a=b$, and $b=c$, then $a=c$ for a,b,c, $\in$ R (Transitivity)

Also we can prove the following properties of real numbers in the field R.

4 If $a,b,c \in R$ and $a=b$, then $a+c=b+c$.
5. If $a,b,c \in R$ and $a=b$, then $ac=bc$.
6 If $a,b,c \in R$ and $a+c=b+c$, then $a=b$

As illustrations, we prove the first two assertions.

**Theorem 1** : For each $a \in R$, $a=a$

**Proof** · Since $a-a=0$,
we have $a=a$ by definition of "=" in R.

**Theorem 2** : If $a,b \in R$ and $a=b$, then $b=a$.
$a=b$.

**Proof** : By the definition of "=" in R,
$a=b$ implies $a-b=0$

Since R is a field, the inverse of $a-b$ under addition i.e. $-(a-b)=b-a \in R$.

But $a-b=0$ and the additive inverse of $0=0$ in R.

So $b-a=0$ as the inverse of element in R is unique.

So $b=a$ from the definition of "=" in R.

Other properties can be taken by the readers as exercises.

Q. 1. Try to give some examples of algebraic structures other than 'group' and 'field'. For example, you may consult some book for definitions of 'ring' and integral domain

Q. 2 Prove the statements 3 to 7 for the field R of all real numbers

Q. 3. Is the converse of the statement 6 true ? Give reasons.

**Solving a Linear Equation in One Variable in the Field R.**

We use the field axioms and properties in solving a simple equation in x. For illustration, consider the following example ·

**Example** : $-2x+5=-7$

**Solution** Adding $-5$ to both sides we have

$-2x+5+(-5)=-7+(-5)$

or $-2x+5-5=-7-5$

or $-2x=-12$

Dividing both sides by $-2$, we get

$x=6$.

We have done a few computations in solving this equation—we added $-5$ to both sides, cancelled $-5$ and 5, divided both sides by $-2$ and reduced the fraction $\frac{-12}{-2}$ to 6.

Now a question arises : what is the rationale behind these computations. These computations are the results of applying field properties of R to the statements during the course of solving the equation. For example adding $-5$ to both sides is justified by the property of equality (=) in R, the field R of real numbers. Whenever no mention is made of any reference field, we take R to be the reference field.

Q. 1. Justify the steps in the solution of the equation $-2x+5=-7$ on basis of properties of the field R.

Q. 2. Solve the following equations for x in R the field of all real numbers and justify your steps—

(a) $x+\frac{2}{3}=\frac{x}{6}-\frac{3}{2}$

(b) $2(x-5)=(2x-4)/5$

Q. 3. If $5x=3x$ for some real number x, is $5=3$ ? What do you conclude from this equation ?

**Teaching Strategies**

To start with, students will find it difficult to understand the rationale of studying algebraic structures because the properties of the numbers are so obvious. To minimise their difficulty the teacher should adopt the following strategies :

The properties of numbers defined by commutative law, associative law, etc., are laws which many a time are satisfied by sets of non-number objects. For example, we can construct a set of colours with an operation acting on colour, e.g., the operation may be defined as mixing one colour b with another colour a so as to produce a third colour c. This can be symbolically represented as $a+b=c$. By giving such examples the teacher can explain that properties of numbers have their counterparts in non-mathematical situations. This understanding will help the students much to visualise that algebraic structures are generalisations of different number systems.

Also care must be taken while defining 'group' and field' to emphasise that the operations mentioned in the definitions need not be usual additions and multiplications. For this purpose, teachers can give examples of the clock-arithmetic and permutation groups. For these examples teachers can consult any standard book on abstract algebra. For lack of space these examples have not been discussed in this lesson.

Teachers should also stress the point that group and field are not the only algebraic structures—there are other types of algebraic structures also. A passing reference to integral domain and ring can be made while teaching this topic.

Students may be made to realise that a study of algebraic structures changes their outlook towards mathematics. For them mathematics becomes a study of systems of objects satisfying certain properties rather than a study of computational procedures ; and they realise that procedures merely help in understanding the properties of the axiom system in which they work.

While explaining the concepts in this topic, teachers should proceed to generalised definitions from particular examples. Along with the examples of concepts, non-examples of the concepts should also be given. This enables the students to be clear about the concepts and know the reasons why a certain object is or is not a case of the concept. This also enables the students to comprehend the finer points of the concepts.

Since most of the properties and concepts in this topic are generalisations of those known earlier to the students, it is better to know what is meant by the generalisation of a concept. The generalisation B of a concept A should satisfy the following requirements :

(a) A is a particular case of B, and (b) B should have at least one particular case other than A. This fact should be illustrated to the students, whenever possible, while teaching this topic.

It should be emphasised that a study of algebraic structures gives us a better understanding of the computational procedures in traditional mathematics. This can be done particularly while teaching, how field properties of R give explanations for the different steps involved in the solution of a simple equation in one variable. Students can be taught how the equation $x^2 - 2 = 0$ does not have a solution in the field of rationals, as its roots are irrationals. It should be stressed that usually equations are solved in R the field of all real numbers, when no reference is made of a reference field.

On the whole, this topic is of new type and quite abstract. So teachers should be very patient with the students and students should be evaluated on a continuous basis while teaching this lesson. If students find any difficulty (generally the difficulty is due to some conceptual misunderstanding) it can be corrected easily by giving proper examples and non-examples of the concept.

The definition of a group given in this lesson, though a correct one, is not the best one from the logical point of view For example, it is assumed as a defining property of a group that the inverse element of any given element of group is unique In fact, we may define a group with less stringent conditions and prove this property of the inverse of any given element of the group. But this suggested approach though more rigorous and more logical from the developmental viewpoint, is abstract and complicated. So this approach has not been followed in this lesson.

**Evaluation**

Some of the behavioural changes which are expected in the students by learning algebraic structures are :

(1) Student can write the axioms of a group and a field.

(2) Student can give examples of the following :

   (a) A commutative group.
   (b) An infinite non-commutative group.
   (c) a field.

(3) Student can solve simple equations in one variable by using the field properties of R the field of all real numbers.

(4) Students can reason out the steps to solve a simple equation in one variable

**Specimen Test Items**

(1) Let the set A consist of two elements a&b. Let (+) and (×) be the two operations in A as given by the following tables.

2nd FACTOR

1st FACTOR

| + | a | b |
|---|---|---|
| a | a | b |
| b | b | a |

Fig. 6.60

2nd FACTOR

1st FACTOR

| X | a | b |
|---|---|---|
| a | a | a |
| b | a | b |

Fig. 6.61

(a) Show that the structure (A, +, ) is field
(b) Write the additive identity.
(c) Write the multiplicative identity.
(d) Write the multiplicative inverse of b

(2) If the structure (A, +, ) is a field and $a \in A$, then prove that

$$a \, . \, 0 = 0$$

where 0 is the additive identity of A.

(3) Give examples of the following group·

(a) Finite non-commutative group.
(b) Infinite non-commutative group.
(c) Finite commutative group.
(d) Group with single element.

(4) Construct the multiplication table for (A) where A = {1, —1, i, —i} and ( . ) is the multiplication of two elements.

(5) Show that $b.a^{-1}$ is the solution of $y.a = b$

(6) Let A = 1, —1 be set. The operation on A is the usual multiplication of A. Verify that A is a group under operation.

(7) Determine whether addition is an operation on each of the following sets :

(a) The set of all multiples of 4 under addition.
(b) ———, —3a, —2a, —0, a, 2a, 3a, ...
where a is not an integer.

(8) Solve $5x - \frac{3}{2} = x + 4$

**Assignments for Teachers**

(1) (a) Is addition a binary operation on the set of odd positive integers ?
(b) For the same set as in (a), is multiplication a binary operation?
(c) Is addition a binary operation on {1, 1, i,—i}?

(2) Study the topics "clock arithmetic" and "the group of the symmetries of an equilateral tringle" from some book on abstract algebra and explain how these are relevant to this lesson.
(3) Explain the importance of the study of algebraic structures in understanding the nature of mathematics.
(4) Is there a group consisting of one element only?
(5) Does the set containing only the number I constitute a group with respect to multiplication as the operation ?
(6) Prove that $(-a) \times (-b) = ab$
(7) Look at the following multiplication table:

| X | I | a | $a^2$ | b | ba | $ba^2$ |
|---|---|---|---|---|---|---|
| I | I | a | $a^2$ | b | ba | $ba^2$ |
| a | a | $a^2$ | I | $ba^2$ | b | ba |
| $a^2$ | $a^2$ | I | a | ba | $ba^2$ | b |
| b | b | ba | $ba^2$ | I | a | $a^2$ |
| ba | ba | $ba^2$ | b | $a^2$ | I | a |
| $ba^2$ | $ba^2$ | b | ba | a | $a^2$ | I |

Fig. 6.62

Is $(1, a, a^2, b, ba, ba^2)$ a commutative group ?
Support your answer.

(8) Look at the factorization

$$x^2 - y^2 = x^2 + xy - xy - y^2$$
$$= x\ (x+y) - y(x+y)$$
$$= (x-y)\ (x+y)$$

x, y are real numbers Write field axioms or properties to justify each step.

(9) Why is the set of integers not a group under subtraction.

(10) Is the set of all sub-sets of { 1, 2, 3 } under "union" of sets ?

(11) If $5x = 3x$, does it follow $5 = 3$ ? What can you deduce from it?

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